Generalizing the Wythoff Game

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1 Introduction

Let c be a positive integer. In the Wythoff Game there are two piles of tokens with two players who alternately take turns making moves. A player may make one of two different moves on each turn. A player can choose to take any positive number of tokens from one pile. A player may also choose to take from both piles, choosing $k(>0)$ from one and $l(>0)$ from the other such that $|k-l|<c$, where c is a fixed positive integer. If a player can not make a move, he or she is the loser. The only restriction of the game is that a player must take at least one token on each turn. There are many different ways of beating an opponent mathematically, and we will present these different techniques and interesting relationships with mathematical proofs we will come up with our own way to generalize the game.

We will start by denoting the different positions that occur in our game. We denote states of the game by ordered pairs (x, y) with $x \leq y$, where x is the number of tokens in one pile and y is the number in the other. Positions in which the Previous player wins no matter what his opponent does are called $P-$positions. On the other hand, positions where the Next player can win no matter what move the opponent makes are called $N-$positions. From these definitions one can see that $(0,0)$ will be a $P-$position for any c because no matter what, the next player cannot move so the Previous player wins. A position $(0, b), b>0$, is always an N-position for every c, because the Next player will use a type-one move to $(0,0)$ and win. If $c=2$, then $(1,3)$ will be a P-position for several reasons.

Say Nathan and Patrick are playing the Wythoff game where $c=2$. There are two piles, 1 token in one pile, and 3 tokens in the other. Thus, we have a $(1,3)$ position. Patrick was the Previous player. We will show that Patrick, unless he makes a dumb move, can win. In other words, Patrick controls his own destiny. If Nathan takes the 1 remaining token from the one pile, Patrick will then be able to take the remaining tokens from the other pile. Similarly, if Nathan were to take 1 from each pile and leave the pile in position $(0,2)$, Patrick can once again take the remaining tokens from the second pile. Also, if Nathan decides that he wants to take the last token from the smaller pile and 2 from the larger pile, then the game will be in position $(0,1)$, and Patrick can simply take the last token and win the game. Nathan can also try taking any amount from the larger pile, but one can see that if he does so, Patrick will be able to take the remaining tokens in both piles with one move! This is why $(1,3)$ is a $P-$position. In general the set of all $P-$positions will be denoted $\mathcal{P}=\{(x, y): x<y, \text{ and } (x, y) \text{ is a } P-\text{position}\}$ and the set of all $N-$positions will be denoted $\mathcal{N}=\{(x, y): x<y, \text{ and } (x, y) \text{ is a } N-\text{position}\}$ [2].

2 Characterization

The first few P-positions for $c=1$ can be seen in Table 1, while the first few P-postitions for the case $c=2$ can be seen in Table 2 [2].

Table 1

n	A_n	B_n
0	0	0
1	1	2
2	3	5
3	4	7
4	6	10
5	8	13
6	9	15
7	11	18
8	12	20
⋮	⋮	⋮

Table 2

n	A_n	B_n
0	0	0
1	1	3
2	2	6
3	4	10
4	5	13
5	7	17
6	8	20
7	9	23
8	11	27
⋮	⋮	⋮

It is interesting to see that in both cases, $B_n - A_n = cn$. Something that is even more interesting is to notice that the entries in each table can be found as follows. Let $A_0 = 0$ and $B_0 = 0$. For $n \geq 1$, define $A_n = \text{mex}\{A_i, B_i : i < n\}$ to be the smallest integer not in the set $\{A_i, B_i : i < n\}$. The term mex stands for *m*inimum *ex*cluded value. Also, define $B_n = A_n + cn$ [2]. It was proved in [2] that the pairs (A_n, B_n) give the winning positions of the Wythoff game. The proof relies on the fact that $\{A_n\}_{n=1}^{\infty}$ and $\{B_n\}_{n=1}^{\infty}$ form complementary sequences.

Definition 1. Two sequences of positive integers, $\{s_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ are complementary if $\{s_n\} \cup \{t_n\} = \mathbb{Z}^+$ and $\{s_n\} \cap \{t_n\} = \emptyset$.

The proof that $A_n = \text{mex}\{A_i, B_i : i < n\}$ and $B_n = A_n + cn$ was given in [2] and uses a simple analysis of the possible scenarios of the game. Later we will prove the generalization of the game using the same method. We now discuss a more elegant, number theoretic proof, also given in [2]. We need the following famous lemma. Recall that $\lfloor x \rfloor$ is the largest integer less than or equal to x.

Lemma 2.1. *Let α and β be positive irrationals satisfying $\alpha^{-1} + \beta^{-1} = 1$. For $n \geq 1$, let $A'_n = \lfloor n\alpha \rfloor$ and $B'_n = \lfloor n\beta \rfloor$. Also let $A' = \cup_{n=1}^{\infty} \{A'_n\}$, and $B' = \cup_{n=1}^{\infty} \{B'_n\}$. Then A' and B' are complementary.*

Proof. It suffices to show that exactly one member of the sequence

$$\psi = \{\alpha, \beta, 2\alpha, 2\beta, 3\alpha, 3\beta, \ldots, n\alpha, n\beta, \ldots\}$$

is in the interval $[h, h+1)$ for every positive integer h. Therefore it suffices to show if $M > 1$ is any integer, then there are exactly $M - 1$ members of ψ less than M. The number n for which $n\alpha < M$ is $\lfloor M/\alpha \rfloor$ and the number n for which $n\beta < M$ is $\lfloor M/\beta \rfloor$. Hence the number of members in ψ less than M is $N = \lfloor M/\alpha \rfloor + \lfloor M/\beta \rfloor$. We know

$$M/\alpha - 1 < \lfloor M/\alpha \rfloor < M/\alpha \text{ and } M/\beta - 1 < \lfloor M/\beta \rfloor < M/\beta.$$

Therefore,

$$\left(\frac{M}{\alpha} - 1\right) + \left(\frac{M}{\beta} - 1\right) < \left\lfloor \frac{M}{\alpha} \right\rfloor + \left\lfloor \frac{M}{\beta} \right\rfloor < \frac{M}{\alpha} + \frac{M}{\beta}$$

$$M\left(\frac{1}{\alpha} + \frac{1}{\beta}\right) - 2 < \left\lfloor \frac{M}{\alpha} \right\rfloor + \left\lfloor \frac{M}{\beta} \right\rfloor < M\left(\frac{1}{\alpha} + \frac{1}{\beta}\right)$$

$$M - 2 < \left\lfloor \frac{M}{\alpha} \right\rfloor + \left\lfloor \frac{M}{\beta} \right\rfloor < M.$$

Thus, $M - 2 < N < M$ and since N is an integer, it follows that $N = M - 1$. □

To use Lemma 2.1 to find the winning positions in the game, consider the quadratic equation $x^2 + (c-2)\,x - c = 0$. The roots of this equation are

$$x = \tfrac{2-c\pm\sqrt{c^2+4}}{2}$$

Let $\alpha = \frac{2-c+\sqrt{c^2+4}}{2}$, the positive root of this equation, and let $\beta = \alpha + c$. Note that $\frac{1}{\alpha} + \frac{1}{\beta} = 1$. The following Theorem was proved in [2].

Theorem 1. *If (A_n, B_n) is a P-position of the Wythoff game, then for $n \geq 0$ $A_n = \lfloor n\alpha \rfloor$ and $B_n = \lfloor n\beta \rfloor$, where $\lfloor x \rfloor$ is the greatest integer less than or equal to x.*

Proof. Note that $A'_0 = \lfloor 0\alpha \rfloor = 0 = A_0$, $B'_0 = 0 = B_0$. Also, $B'_n = A'_n + cn$, since $A'_n = \lfloor n\alpha \rfloor$ and

$$\begin{aligned} B'_n &= \lfloor n\beta \rfloor \\ &= \lfloor n\,(\alpha + c) \rfloor \\ &= \lfloor n\alpha + nc \rfloor \\ &= \lfloor n\alpha \rfloor + nc, \quad \text{since} \quad nc \in \mathbb{Z} \\ &= A'_n + cn. \end{aligned}$$

All that remains is to show that A'_n =mex$\{A_i, B_i : i < n\}$. By definition, $\{A'_n\}_{n=1}^{\infty}$ and $\{B'_n\}_{n=1}^{\infty}$ are increasing sequences. By Lemma 2.1, these sequences are complementary. Thus, if A'_n were not mex$\{A'_i, B'_i\}$, then A'_i would never be obtained in $\{A'_n\} \cup \{B'_n\}$. □

We will now give another means for determining the winning positions in the game. This characterization of the winning positions will enable us to generalize the game. First we need some definitions.

3 Morphisms

The definitions provided in this section were found in [1] and [3]. Recall a group is a nonempty set G with a binary operation in which

$i)$ the operation is associative.
$ii)$ every element of G has an identity element.
$iii)$ every element of G has an inverse under the operation.

A monoid is a set M with a binary operation in which

$i)$ the operation is associative.
$ii)$ every element of G has an identity element.

For example, let A be a finite set, say $A = \{a, b\}$. Let A^* be set of all finite words over A. Thus

$$A^* = \{\lambda, a, b, aa, ab, ba, bb, aaa, aab, \cdots\}$$

where λ is the empty word. We make A^* into a monoid under the operation of concatenation. In other words, if $x, y \in A^*$, then $xy \in A^*$.

The following definitions are found in [1].

Definition 2. Let A be a finite set. An infinite word over A is an infinite, ordered sequence over A. We write an infinite word $w = \{x_i\}_{i=1}^{\infty}$ using concatenation of the symbols, $w = x_1 x_2 x_3 \ldots$, where $x_i \in A$ for all i.

Definition 3. Let M be a monoid, and let $\sigma : M \to M$ be a mapping on M with the property

$$\sigma(xy) = \sigma(x)\,\sigma(y) \text{ for all } x, y \in M.$$

Then σ is a morphism of M.

Definition 4. Let A be a finite set, and let $\{w_n\}$ be an infinite sequence of words in A^*. Let w be an infinite word over A. Then w_n converges to w, written $w_n \to w$, if for every prefix u of w there exists an N such that $n \geq N$ implies that u is a prefix of w_n.

As an example of this definition, consider the sequence $\{u_n\}$, where

$$\begin{aligned} u_1 &= a \\ u_2 &= ab \\ u_3 &= aba \\ u_4 &= abab \end{aligned}$$

and in general,

$$u_n = \begin{cases} (ab)^{\frac{n}{2}} & \text{if } n \text{ is even} \\ (ab)^{\frac{n-1}{2}} a & \text{if } n \text{ is odd.} \end{cases}$$

Clearly $\lim_{n\to\infty} u_n = ababababab\cdots$.

We now give an example of a morphism. Define $\sigma : A^* \to A^*$ by first defining $\sigma(a) = ab$ and $\sigma(b) = a$. We then extend σ to all of A^* by the rule

$$\sigma(xy) = \sigma(x)\,\sigma(y).$$

So, for example,

$$\sigma(ab) = \sigma(a)\,\sigma(b) = aba.$$

and

$$\sigma(abb) = \sigma(ab)\,\sigma(b) = abaa.$$

By definition, σ is clearly a morphism on the monoid A^*. The morphism in the previous example is known as the Fibonacci morphism [1]. The reason it is called the Fibonacci morphism is as follows. Start with a, and iterate σ on a repeatedly, using the notation that

$$\sigma^n(x) = \sigma(\sigma(\sigma(\cdots(\sigma(x)\cdots)))).$$

where the composition is performed n times. Define $w_n = \sigma^n(a)$ for $n \geq 0$. Then the first few w_n are

$$w_0 = \sigma^0(a) = a$$
$$w_1 = \sigma^1(a) = ab$$
$$w_2 = \sigma^2(a) = \sigma(\sigma(a)) = \sigma(ab) = \sigma(a)\sigma(b) = aba$$
$$w_3 = \sigma^3(a) = \sigma\left(\sigma^2(a)\right) = \sigma(aba) = abaab$$
$$w_4 = abaababa$$

Consider the infinite sequence $w_0,\ w_1,\ w_2,\ \ldots$. Letting n increase indefinitely, we arrive at an infinite word known as the Fibonacci word. The first few characters of this word are

$$abaababaabaab\cdots$$

The Fibonacci word can now be defined as

$$\lim_{n\to\infty} \sigma^n(a)$$

where σ is the Fibonacci morphism. It makes sense to do this, because the sequence $\{\sigma^n(a)\}_{n=0}^{\infty}$ converges. To see why, notice that

$$\begin{aligned} \sigma^{n+1}(a) &= \sigma^n(\sigma(a)) \\ &= \sigma^n(ab) \\ &= \sigma^n(a)\sigma^n(b) \end{aligned}$$

Thus, $\sigma^n(a)$ is a prefix of $\sigma^{n+1}(a)$, so there will be an infinite word that satisfies definition 4.

Ex: Let $A = \{a, b\}$. Define $\sigma(a) = aab$ and $\sigma(b) = a$. What is the $\lim_{n\to\infty} \sigma^n(a)$?

$$\sigma^0(a) = a$$
$$\sigma^1(a) = aab$$
$$\sigma^2(a) = \sigma(\sigma(a)) = \sigma(aab) = \sigma(a)\sigma(a)\sigma(b) = aabaaba$$
$$\sigma^3 = \sigma(\sigma^2(a)) = \sigma(aabaaba) = \sigma(a)\sigma(a)\sigma(b)\sigma(a)\sigma(a)\sigma(b)\sigma(a) = aabaabaaabaabaaab$$

Again, as in the Fibonacci word, we see that $\sigma^n(a)$ is a prefix of $\sigma^{n+1}(a)$ for every n. Thus $\sigma^n(a)$ converges to some limit. The first few characters of the limit are

$$\lim_{n\to\infty} \sigma^n(a) = aabaabaaabaabaaabaabaabaaabaabaaabaabaaba\cdots$$

Ex: Let $A = \{a, b\}$. Define $\sigma(a) = aab$ and $\sigma(b) = aa$. What is $\lim_{n\to\infty} \sigma^n(a)$?

$$\sigma^0(a) = a$$
$$\sigma^1(a) = aab$$
$$\sigma^2(a) = \sigma(\sigma(a)) = \sigma(aab) = \sigma(a)\sigma(a)\sigma(b) = aabaabaa$$
$$\sigma^3(a) = \sigma(\sigma^2(a)) = \sigma(aabaabaa) = \sigma(a)\sigma(a)\sigma(b)\sigma(a)\sigma(a)\sigma(b)\sigma(a)\sigma(a) = aabaabaaaabaabaaaabaab$$

Again, we see that $\sigma^n(a)$ is a prefix of $\sigma^{n+1}(a)$. So $\sigma^n(a)$ converges. The first few characters of the limit are

$$\lim_{n\to\infty} \sigma^n(a) = aabaabaaaabaabaaaabaabaabaabaaaaabaabaaaabaabaabaabaaaabaabaa\cdots$$

This example will later be used in generalizing the Wythoff game.

4 Fibonacci Word

In [2] it was noted that the Fibonacci Word and related morphisms give us another way to determine the winning positions in the Wythoff game. Consider the positions of the a's and b's in the Fibonacci word.

n	1	2	3	4	5	6	7	8	9	10	11	12	$\cdots$
	a	b	a	a	b	a	b	a	a	b	a	a	$\cdots$

The a's occur in positions $1, 3, 4, 6, 8, 9, 11, 12, \ldots$, which are precisely the values $\lfloor n\alpha \rfloor$, where $\alpha = \frac{1+\sqrt{5}}{2}$ (see chapter 9 of [1]). Also, the b's occur in positions $2, 5, 7, 10, \ldots$ which are the complements of $\{\lfloor n\alpha \rfloor\}$. If we put these sequences side by side in a column, then by Theorem 1 we get the winning positions in the case that $c = 1$,

a positions	b positions
1	2
3	5
4	7
6	10
8	13
9	15
$\vdots$	$\vdots$

For the Wythoff game with $c = 2$ we consider the word generated by the morphism $\sigma(a) = aab$, $\sigma(b) = a$. This word is

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	$\cdots$
	a	a	b	a	a	b	a	a	a	b	a	a	b	a	$\cdots$

Lining up the positions of the a's and b's side by side in columns gives the winning positions of the game for $c = 2$. This follows from Theorem 1 and the fact that the positions of the a's are given by the sequence $\lfloor n\alpha \rfloor$, with $\alpha = \sqrt{2}$. The positions of the b's are given by the sequence $\lfloor n\beta \rfloor$, with $\beta = \sqrt{2} + 2$. (See chapter 9 of [1]).

a positions	b positions
1	3
2	6
4	10
5	13
7	17
8	20
9	23
$\vdots$	$\vdots$

In general, if $c > 1$, then the winning positions of the game are given by the positions of the a's and b's in the infinite word generated by the morphism $\sigma(a) = a^c b$, $\sigma(b) = a$ [2].

5 Generalizing the Game

This last characterization of the winning positions of the game as the positions of the a's and b's in the infinite words generated by morphisms suggests a way to generalize the game. The idea is to start with an infinite word generated by some morphism and look at the positions of the a's and b's in that word. We then try to come up rules to a similar game having these positions as winning positions. This is not easy to do in general, but there is a class of morphisms similar to the morphisms in the previous section. For example, consider

$$\sigma(a) = aab$$

$$\sigma(b) = aa.$$

The word that is generated is $aabaabaaaabaabaaaabaabaabaab\ldots$. So we want the winning positions to be

Table 3

n	Positions of a's	Positions of b's
1	1	3
2	2	6
3	4	11
4	5	14
5	7	19
6	8	22
7	9	25
8	10	28
⋮	⋮	⋮

For the remainder of this section let A_n be the position of the n^{th} a in the infinite word, and let B_n be the position of the n^{th} b. Notice that that $A_n = mex\,\{A_i, B_i\}$, and $B_n = 2A_n + n$. We then came up with the following rules to a new game.
i) A player can remove any number from a single pile.
ii) A player can remove k from one pile, and l from another. If $k < l$, then we must have $k \leq l \leq 2k$.

Suppose now that Nathan and Patrick are playing this new version of the game. There are two piles, 2 tokens in one pile and 6 tokens in the other. Thus, we have a (2,6) position. Patrick was the Previous player and we will show that unless he makes a dumb move, he can win. If Nathan takes 1 from the smaller pile the position goes to (1,6) and then it is Patrick's turn. So then Patrick would remove 3 tokens from the larger pile to get to the position (1,3). From this position Nathan cannot win because the most he can remove is 1 token from one pile and 2 tokens from the other, which would leave one remaining token for Patrick to take. If Nathan removes any other amount from either pile, Patrick will be able to take the rest of the tokens in one turn because the amount he takes from one pile will be less than or equal to twice what he takes from the other. Now a similar senario follows if from the start Nathan takes 1 token from one pile and 1 from the other or 1 from one pile and 2 from another because this would lead to positions of (1,5) and (1,4), respectively. From these positions Patrick can once again get to the position (1,3). Say Nathan removes 1 token from the larger pile to move to the positon (2,5), then Patrick can move to the position (1,3) and the same follows from before and Patrick wins. If Nathan were to move to the position (2,4) by removing 2 tokens from the larger pile, Patrick can take all of the remaining tokens since $4 \leq 2(2)$.

In a similar way if Nathan makes any other move to any other possible position, Patrick will still win. Therefore, no matter what Nathan does, Patrick wins, so (2,6) is a winning position.

Theorem 2. *Let* $A_0 = 0$*,* $B_0 = 0$*. For* $n \geq 1$*, define* $A_n =$*mex*$\{A_i, B_i : 0 \leq i \leq n\}$ *and* $B_n = n + 2A_n$*. Let* $A = \{A_n\}_{n=1}^{\infty}$ *and* $B = \{B_n\}_{n=1}^{\infty}$*. Then* A *and* B *are complementary, that is,* $A \cap B = \emptyset$ *and* $A \cup B = \mathbb{Z}^+$*.*

Proof. First show $A \cap B = \emptyset$. Suppose that there exists $x \in A \cap B$. Then $x \in A$ and $x \in B$. Since $x \in A$, then $x = A_n$ for some n. Since $x \in B$, $x = B_m$ for some m. By definition of B_m, $A_n = x = B_m = 2A_m + m$. But this implies that $A_n = B_m = 2A_m + m > A_m$. Since $A_n =$mex$\{A_i, B_i : 0 \leq i \leq n\}$, this means that $m < n$, since $A_n > A_m$. However, since $A_n = B_m$, this would imply that $B_m \notin \{A_i, B_i : 0 \leq i \leq n\}$ and $m > n$, which is a contradiction. □

Theorem 3. *If* $\mathcal{P}$ *is the set of winning positions in the new game, then* $\mathcal{P} = \cup_{i=0}^{\infty} \{(A_i, B_i)\}$ *where* $A_0 = 0$*,* $B_0 = 0$*,* $A_n =$*mex*$\{A_i, B_i\}$ *and* $B_n = 2A_n + n$*.*

We will now prove this theorem. Our proof to this theorem is similar to the one for the Wythoff Game in [2].

Proof. We need to show two things.
(I) A player moving from some (A_n, B_n) lands in a position not of the form (A_i, B_i).
(II) Given any position$(x, y) \neq (A_i, B_i)$, there is some move to a positon of the form (A_n, B_n).
We first prove (I).
I. It is clear that a move of the first type from an (A_n, B_n) position is not in an (A_i, B_i) position. Suppose a move of the second type from (A_n, B_n) results in (A_i, B_i). Then $i < n$. Also, there are $B_n - B_i$ tokens removed from the B pile, and $A_n - A_i$ tokens removed from the A pile. So we must have either

$$B_n - B_i \leq A_n - A_i \leq 2(B_n - B_i)$$

if more tokens are taken from pile A, or

$$A_n - A_i \leq B_n - B_i \leq 2(A_n - A_i)$$

if more are taken from the B pile. In the first case, we have

$$2A_n + n - (2A_i + i) \leq A_n - A_i \leq 2(2A_n + n - 2A_i - i).$$

So,

$$2A_n - 2A_i + n - i \leq A_n - A_i \leq 4A_n - 4A_i + 2n - 2i.$$

The first inequality implies $A_n - A_i + n - i \leq 0$. However, since $i < n$ and $A_i < A_n$, this is a contradiction. In the second case, we have

$$A_n - A_i \leq 2A_n + n - 2A_i - i \leq 2A_n - 2A_i$$

But again, the second inequality implies $n - i \leq 0$, a contradiction.

II. Let (x, y), with $x \leq y$, be a position not of the form (A_i, B_i), $i \leq 0$. Since A and B are complementary, either $x = A_n$ or $x = B_n$ for some $n \geq 0$. If $x = B_n$, simply move y to A_n. Suppose that $x = A_n$. If $y > B_n$, move y to B_n. Suppose then that $A_n \leq y < B_n$. If $A_n \leq y \leq 2x$, then move (x, y) to $(0, 0)$. This is clearly a legal move. Suppose then that $2x < y < B_n$. Then we need to find an m such that we can move (x, y) to (A_m, B_m).

If we can find m such that

$$x - A_m = A_n - A_m \leq y - B_m \leq 2(A_n - A_m) = 2(x - A_m)$$

then we will have a legal move. Simplifying the second inequality above, we need m such that

$$y - 2A_m - m = y - B_m < 2A_n - 2A_m$$

Thus, we need $y - 2A_m - m < 2A_n - 2A_m$ or $y - m < 2A_n = 2x$. So let $m = y - 2x$. We automatically have that $y - B_m \leq 2(A_n - A_m) = 2(x - A_m)$ is satisfied. We only need to verify that $x - A_m = A_n - A_m \leq y - B_m$. Notice that

$$\begin{aligned} y - B_m &= y - (2A_m + m) \\ &= y - (2A_m + y - 2x) \\ &= y - 2A_m - y + 2x \\ &= 2x - 2A_m \\ &= 2(x - A_m) \\ &= 2(A_n - A_m), \text{ since } x = A_n \\ &> A_n - A_m, \text{ since } A_n - A_m > 0. \end{aligned}$$

So moving (x, y) to (A_m, B_m) is legal. □

From this final proof, we have generalized the Wythoff game. By finding new rules that differ from the original Wythoff game, exploring the different aspects of the Wythoff game, and by seeing how the Wythoff game works, we have found all of the winning positions of the new game. The first few winning positions can be seen in Table 3 and by Theorem 3, the remaining winning positions can be found.

References

[1] Jean-Paul Allouche and Jeffrey Shallit, *Automatic Sequences: Theory, Applications, Generalizations*, Cambridge University Press, Cambridge, 2003.

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[3] M. Lothaire, *Algebraic Combinatorics on Words*, Cambridge University Press, Cambridge, 2002.