# The Difference Sequences of the Columns of Wythoff's Array

Halleh Peterman
Advisor: Dr. David Garth

## 1 Introduction

The Fibonacci numbers are defined by $F_0 = 0, F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$ . For the remainder of this paper, let $\phi = \frac{1+\sqrt{5}}{2}$. The Wythoff array, $W$, was first introduced by David R. Morrison [1] in 1980 as

$$w_{i,1} = \lfloor \lfloor i\phi \rfloor \phi \rfloor$$
$$w_{i,2} = \lfloor \lfloor i\phi \rfloor \phi^2 \rfloor,$$

where $w_{i,1}$ represents row $i$ column 1 and $w_{i,2}$ represents row $i$ column 2. Recall that the floor of a number $x$ is the largest integer not exceeding $x$. The subsequent entries in each row are then generated by the Fibonacci recurrence.

*Example* 1. For row 3:

$$\lfloor \lfloor 3\phi \rfloor \phi \rfloor = 6$$
$$\lfloor \lfloor 3\phi \rfloor \phi^2 \rfloor = 10$$

So the third row is:

6,10,16,26,42,68...

Morrison [1] proved that the first row of the Wythoff array is the Fibonacci numbers and the first column of the array is computed by the formula

$$w_{i,1} = \lfloor i\phi \rfloor + i - 1,$$

After finding the first number of a row by the above formula, Morrison proved that the rest of $W$ can be computed using the formula

$$w_{i,j+1} = \begin{cases} \lfloor \phi w_{ij} \rfloor + 1 & \text{if } j \text{ is odd} \\ \lfloor \phi w_{ij} \rfloor & \text{if } j \text{ is even} \end{cases} \quad \text{for } i, j = 1, 2, 3, \ldots \tag{1}$$

where $w_{ij}$ is row $i$ column $j$ of $W$. Note that the first row of the Wythoff Array is $1, 2, 3, 5, 8, ...$ which is $F_2, F_3, F_4, F_5, F_6, ...$, so the entry $w_{1,j} = F_{j+1}$. The rest of the Wythoff array is as follows.

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 5 | 8 | 13 | 21 | 34 | 55 | 89 | 144 | ... |
| 4 | 7 | 11 | 18 | 29 | 47 | 76 | 123 | 199 | 322 | 521 | |
| 6 | 10 | 16 | 26 | 42 | 68 | 110 | 178 | 288 | 466 | 754 | |
| 9 | 15 | 24 | 39 | 63 | 102 | 165 | 267 | 432 | 699 | 1131 | |
| 12 | 20 | 32 | 52 | 84 | 136 | 220 | 356 | 576 | 932 | 1508 | |
| 14 | 23 | 37 | 60 | 97 | 157 | 254 | 411 | 665 | 1076 | 1741 | |
| 17 | 28 | 45 | 73 | 118 | 191 | 309 | 500 | 809 | 1309 | 2118 | |
| 19 | 31 | 50 | 81 | 131 | 212 | 343 | 555 | 898 | 1453 | 2351 | |
| 22 | 36 | 58 | 94 | 152 | 246 | 398 | 644 | 1042 | 1686 | 2728 | |
| ⋮ | | | | | | | | | | | |

There are a lot of interesting structures in this array [3]. For this paper, we will look at each column and examine the differences between subsequent entries in the columns:

$$\text{Column 1:}\quad 1\underbrace{\quad}_{3}4\underbrace{\quad}_{2}6\underbrace{\quad}_{3}9\underbrace{\quad}_{3}12\underbrace{\quad}_{2}14\underbrace{\quad}_{3}17\underbrace{\quad}_{2}19\underbrace{\quad}_{3}22$$

$$\text{Column 2:}\quad 2\underbrace{\quad}_{5}7\underbrace{\quad}_{3}10\underbrace{\quad}_{5}15\underbrace{\quad}_{5}20\underbrace{\quad}_{3}23\underbrace{\quad}_{5}28\underbrace{\quad}_{3}31\underbrace{\quad}_{5}36$$

$$\text{Column 3:}\quad 3\underbrace{\quad}_{8}11\underbrace{\quad}_{5}16\underbrace{\quad}_{8}24\underbrace{\quad}_{8}32\underbrace{\quad}_{5}37\underbrace{\quad}_{8}45\underbrace{\quad}_{5}50\underbrace{\quad}_{8}58$$

$$\text{Column 4:}\quad 5\underbrace{\quad}_{13}18\underbrace{\quad}_{8}26\underbrace{\quad}_{13}39\underbrace{\quad}_{13}52\underbrace{\quad}_{8}60\underbrace{\quad}_{13}73\underbrace{\quad}_{8}81\underbrace{\quad}_{13}94$$

Now let's compare these difference squences in the columns next to one another:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 3 | 2 | 3 | 3 | 2 | 3 | 2 | 3 |
| 5 | 3 | 5 | 5 | 3 | 5 | 3 | 5 |
| 8 | 5 | 8 | 8 | 5 | 8 | 5 | 8 |
| 13 | 8 | 13 | 13 | 8 | 13 | 8 | 13 |

By looking at these column differences, we notice a pattern where if the first number is $a$ and the second number is $b$, then the pattern of the column differences is as follows:

$$a \quad b \quad a \quad a \quad b \quad a \quad b \quad a \quad \ldots$$

Also, moving to the next row moves $a$ and $b$ to the next Fibonacci numbers. This pattern is actually a very famous pattern known as the Fibonacci word. Dr. Martin Erickson brought this result, without proof, to the attention of my capstone advisor, Dr. David Garth. Dr. Garth then contacted Clark Kimberling[4], perhaps the leading expert on the Wythoff array, about the result. Clark Kimberling responded by saying that he did not recall seeing this result published anywhere. We will later prove that this pattern holds for all columns of the Wythoff array. First we need to discuss the theory of infinite words.

## 2 Monoids, Morphisms, and the Fibonacci Word

Recall that a group is a nonempty set together with a binary operation that is associative and has an identity in which every element in the set has an inverse under the operation. If we remove the requirement that every element has an inverse, then we get a monoid [5].

**Definition 1.** A monoid is a set $M$ together with a binary operation such that the following hold:

1. $x(yz) = (xy)z$ for al $x, y, z \in M$
2. There is a $\lambda \in M$ such that $x\lambda = \lambda x = x$ for all $x \in M$.

*Example* 2. Let $M = \{0,1,2,3, ...\}$, the set of nonnegative integers under addition. Then $M$ is a monoid.

*Example* 3. Let $A = \{a, b\}$, and let $A^*$ be the set of finite words over A, with $\lambda$ being the empty word.

$$A^* = \{\lambda, a, b, aa, ab, ba, bb, aaa, aab, \ldots\},$$

So $A^*$ is a monoid, where the operation is concatenation of words. So, for example, if $x = ab$ and $y = baa$ then $xy = abbaa$.

**Definition 2.** A morphism of a monoid $M$ is a map $\sigma : M \to M$ such that $\sigma(xy) = \sigma(x)\sigma(y)$ for all $x, y \in M$.

*Example* 4. The map $\sigma : \mathbb{Z}^+ \cup \{0\} \to \mathbb{Z}^+ \cup \{0\}$ defined by $\sigma(n) = 2n = n + n$ is a morphism of the monoid of nonnegative integers where the operation on integers is addition. This is because

$$\begin{aligned}\sigma(m+n) &= 2(m+n) \\ &= 2m + 2n \\ &= \sigma(m) + \sigma(n).\end{aligned}$$

*Example* 5. Let $A = \{a, b\}$ and define $\sigma$: $A^* \to A^*$ by first letting $\sigma(a) = ab$ and $\sigma(b) = a$. We can extend $\sigma$ to a morphism on $A^*$ by concatination. In other words, if $x, y \in A^*$, and if $\sigma(x)$ and $\sigma(y)$ are defined, then $\sigma(xy)$ is defined to be $\sigma(x)\sigma(y)$. We also define $\sigma(\lambda) = \lambda$. So, for example:

$$\begin{gathered}\sigma(ab) = \sigma(a)\sigma(b) = aba \\ \sigma(ba) = \sigma(b)\sigma(a) = aab \\ \sigma(bb) = \sigma(b)\sigma(b) = aa \\ \sigma(aa) = \sigma(a)\sigma(a) = abab\end{gathered}$$

Continuing in this manner, we extend $\sigma$ to a map on all of $A^*$. Thus $\sigma$ is clearly a morphism. This morphism is known as the Fibonacci morphism [2].

Our next goal is to define the Fibonacci word, which is the word discussed at the end of the introduction. To do this, we use the Fibonacci morphism. First we need to define the notion of an infinite word as a limit of an infinite sequence of finite words.

**Definition 3.** For $A = \{a, b\}$, take a sequence of words $\{u_n\}$, where

$$u_1, u_2, u_3, \ldots \in A^*.$$

Then $\{u_n\}$ converges to the infinite word $u$, written $\lim_{n\to\infty} u_n = u$, if and only if every prefix of $u$ is a prefix of all but finitely many $u_n$'s.

*Example* 6. Let:

$$\begin{aligned}u_1 &= a \\ u_2 &= ab \\ u_3 &= aba \\ u_4 &= abab \\ u_5 &= ababa\end{aligned}$$

Clearly under this definition, $u_n$ converges to the infinite word $u = abababab\ldots$. For example, consider the prefix $ababab$ of $u$. Then $ababab$ is a prefix of all $u_n$'s except $u_1, ..., u_5$.

We now define the Fibonacci word. From now on, let $\sigma$ be the Fibonacci morphism mentioned earlier. The idea is to iterate $\sigma$ on $a$. We let $u_n = \sigma^n(a)$, where the exponential notation refers to function composition.

$$\begin{aligned} u_0 &= \sigma^0(a) = a \\ u_1 &= \sigma^1(a) = ab \\ u_2 &= \sigma^2(a) = \sigma(\sigma(a)) = \sigma(ab) = \sigma(a)\sigma(b) = aba \\ u_3 &= \sigma^3(a) = \sigma(\sigma(\sigma(a))) = \sigma(aba) = \sigma(a)\sigma(b)\sigma(a) = abaab \end{aligned}$$

Since compositions of morphisms are morphisms, notice that each $u_n$ is a prefix of $u_{n+1}$. To see why, notice that $u_{n+1} = \sigma^{n+1}(a) = \sigma^n(\sigma(a)) = \sigma^n(ab) = \sigma^n(a)\sigma^n(b) = u_n\sigma^n(b)$. So by definition, $\sigma^n(a)$ clearly converges to some infinite word $z$, and we define this word $z$ to be the Fibonacci word. The first few characters in $z$ are as follows,

$$z = \lim_{n\to\infty} \sigma^n(a) = abaababaabaababaababa\ldots$$

Later we will need to extend the Fibonacci morphism $\sigma$ to infinite words. We do this as follows. If

$$x = x_1x_2x_3\ldots$$

is an infinite word, with $x_i \in \{a, b\}$, then we define $\sigma(x)$ to be the word

$$\sigma(x_1)\sigma(x_2)\sigma(x_3)\ldots$$

We will be using the Fibonacci word to prove our main result about the Wythoff array. However, it is worth mentioning some other interesting properties of the word. Proofs of these properties can be found in [2] and [5]. For the following examples, the Fibonacci word will be denoted as $z$.

**Properties of the Fibonacci Word**

1. The Fibonacci word is a fixed point of $\sigma$ [5]. In other words, $\sigma(z) = z$. This seems plausible, since

$$\begin{aligned} \sigma(abaababaab\ldots) &= \sigma(a)\sigma(b)\sigma(a)\ldots \\ &= abaababaab\ldots \end{aligned}$$

   Moreover, the Fibonacci word is the only word, finite or infinite, with this property ([5], section 1.2).

2. $\sigma^n(a) = \sigma^{n-1}(a)\sigma^{n-2}(a)$, which tells us that $u_n = u_{n-1}u_{n-2}$ ([2], chapter 7). Each prefix $u_n$ is the concatenation of the previous two words $u_{n-1}$ and $u_{n-2}$. The word can be generated by initial conditions $u_1 = a$, $u_2 = ab$ as follows:

$$\begin{aligned} u_1 &= a \\ u_2 &= ab \\ u_3 &= \underbrace{ab}\,\underbrace{a} \\ u_4 &= \underbrace{aba}\,\underbrace{ab} \\ u_5 &= \underbrace{abaab}\,\underbrace{aba} \\ u_6 &= \underbrace{abaababa}\,\underbrace{abaab} \\ &\vdots \end{aligned}$$

   If we change the intial conditions, we get a different word using the same operation.

3. The length of $u_n$ is $F_{n+1}$, the number of $a$'s in $u_n$ is $F_n$ and the number of $b$'s in $u_n$ is $F_{n-1}$ ([2] chapter 7).
4. For every $n$, the Fibonacci word contains $n+1$ subwords of length $n$ ([5], chapter 6). The following table illustrates this:

| $n$ | Subwords of length n |
|---|---|
| 1 | $a, b$ |
| 2 | $aa, ab, ba$ |
| 3 | $aba, aab, bab, baa$ |

5. The last two characters of each $u_n$ alternate between $ab$ and $ba$ ([2], chapter 7):

$$\underline{ab}$$
$$a\underline{ba}$$
$$aba\underline{ab}$$
$$abaaba\underline{ba}$$
$$abaababaaba\underline{ab}$$
$$abaababaabaababaababa\underline{ba}$$

We also observe that removing the last two characters from each $u_n$ gives a palindrome ([2], chapter 7).

6. The Fibonacci word is nonperiodic, meaning that no portion of the word repeats itself.
7. For any two factors of the Fibonacci word of the same length, the difference in the number of $a$'s in each factor is less than or equal to 1 ([6], chapter 2).

*Example* 7. *baabab*, characters 2-7 of the Fibonacci word, contain three $a$'s. *aabaab*, characters 8-13, contain four $a$'s.
$4 - 3 = 1 \leq 1$

*Example* 8. *baab*, characters 2-5 of the Fibonacci word, contain two $a$'s. *baba*, characters 18-21, contain two $a$'s.
$2 - 2 = 0 \leq 1$

# 3 Main Theorem

The definitions in the last section enable us to state our main result noticed in the introduction. That is, the sequence $abaababaab\ldots$ is the pattern noticed in the difference sequences of the columns.

**Main Theorem.** Column $j$ of the Wythoff array, $W$, has a difference sequence given by the Fibonacci word with $a = F_{j+3}$ and $b = F_{j+2}$.

We will now prove this. Recall that $w_{1j}$, the $j^{th}$ entry in row one of the array is

$$w_{1j} = F_{j+1},$$

the $(j+1)^{st}$ Fibonacci number. Our proof relies on the following formula [3]:

$$w_{ij} = \lfloor i\phi \rfloor F_{j+1} + (i-1)F_j \text{ for } i, j \geq 1 \tag{2}$$

for the entry in row $i$, column $j$ of the array. We were unable to find a proof of this result. We came to believe it could be proved using methods of [4], but that the proof would be messy and tedious. We are interested in the difference sequence of the columns. That is, we are interested in the sequences

$$w_{i+1,j} - w_{ij}$$

where $j \geq 1$ is the index of a fixed column and $i \geq 1$. Notice that by (2) we have

$$\begin{aligned} w_{i+1,j} - w_{ij} &= (\lfloor (i+1)\phi \rfloor F_{j+1} + iF_j) - (\lfloor i\phi \rfloor F_{j+1} + (i-1)F_j) \\ &= \lfloor (i+1)\phi \rfloor F_{j+1} + iF_j - \lfloor i\phi \rfloor F_{j+1} - (i-1)F_j \\ &= \lfloor (i+1)\phi \rfloor F_{j+1} - \lfloor i\phi \rfloor F_{j+1} + iF_j - iF_j + F_j \\ &= (\lfloor (i+1)\phi \rfloor - \lfloor i\phi \rfloor)F_{j+1} + F_j. \end{aligned} \tag{3}$$

We will show that $\lfloor (i+1)\phi \rfloor - \lfloor i\phi \rfloor$ is always either 1 or 2. It will follow then that either

$$w_{i+1,j} - w_{ij} = \begin{cases} F_{j+1} + F_j = F_{j+2} & \text{if } \lfloor (i+1)\phi \rfloor - \lfloor i\phi \rfloor = 1 \\ 2F_{j+1} + F_j = F_{j+1} + F_{j+1} + F_j = F_{j+1} + F_{j+2} = F_{j+3} & \text{if } \lfloor (i+1)\phi \rfloor - \lfloor i\phi \rfloor = 2 \end{cases} \tag{4}$$

Suppose for now that we have shown that $\lfloor (i+1)\phi \rfloor - \lfloor i\phi \rfloor$ is always either 1 or 2. Let us also consider the pattern of the sequence $\lfloor (i+1)\phi \rfloor - \lfloor i\phi \rfloor$. First notice that:

$$\begin{aligned} \lfloor \phi \rfloor &= 1 \\ \lfloor 2\phi \rfloor &= 3 \\ \lfloor 3\phi \rfloor &= 4 \\ \lfloor 4\phi \rfloor &= 6 \\ \lfloor 5\phi \rfloor &= 8 \\ \lfloor 6\phi \rfloor &= 9 \\ \lfloor 7\phi \rfloor &= 11 \\ \lfloor 8\phi \rfloor &= 12 \\ \lfloor 9\phi \rfloor &= 14 \\ \lfloor 10\phi \rfloor &= 16 \end{aligned}$$

Notice also the difference between the consecutive $\lfloor i\phi \rfloor$:

$$1 \underbrace{\quad}_{2} 3 \underbrace{\quad}_{1} 4 \underbrace{\quad}_{2} 6 \underbrace{\quad}_{2} 8 \underbrace{\quad}_{1} 9 \underbrace{\quad}_{2} 11 \underbrace{\quad}_{1} 12 \underbrace{\quad}_{2} 14 \underbrace{\quad}_{2} 16$$

Again, we see the pattern of the Fibonacci word, in this case with $a = 2$ and $b = 1$. We will show that this pattern holds. To summarize where we are going, we will show that

$$w_{i+1,j} - w_{ij}$$

is the Fibonacci word, with $a = F_{j+3}$, $b = F_{j+2}$, by showing that $\lfloor (i+1)\phi \rfloor - \lfloor i\phi \rfloor$ is the Fibonacci word with $a = 2$, $b = 1$. To do this, we will show that the word $f_1 f_2 f_3 \ldots$, with $f_i$ defined by

$$f_i = \lfloor (i+1)\alpha \rfloor - \lfloor i\alpha \rfloor$$

gives the Fibonacci word with $a = 1$, $b = 0$. Finally, in order to do this, we consider the word $g_1 g_2 g_3 \ldots$ with $g_n$ defined by

$$g_n = \begin{cases} 1 & \text{if } n = \lfloor k\phi \rfloor \text{ for some integer k} \\ 0 & \text{otherwise} \end{cases}$$

and show $f_n = g_n$ for every $n$. We will then show that $\sigma(f) = g = f$, where $\sigma$ is the Fibonacci morphism. It will follow from the first property of the Fibonacci word mentioned in Section 2 that $f$ is the Fibonacci word.

Thus, we start by defining

$$v_i = \lfloor (i+1)\phi \rfloor - \lfloor i\phi \rfloor$$

and the infinite word

$$v = v_1 v_2 v_3 v_4 v_5 ... = 21221...$$

Also, let $y$ be the infinite word $y_1 y_2 y_3 \ldots$, where

$$y_i = w_{i+1,j} - w_{ij}.$$

Note that by (4),

$$y_i = \begin{cases} F_{j+3} & \text{if } v_i = 2 \\ F_{j+2} & \text{if } v_i = 1 \end{cases}$$

Thus, if we can show that $v$ is the Fibonacci word with $a = 2$ and $b = 1$, then $y$ will be the Fibonacci word with $a = F_{j+3}$ and $b = F_{j+2}$. This is precisely what we wish to prove. Our strategy will actually involve consideration of the following sequence from chapter 9 of [2]. Let $\alpha = \frac{\sqrt{5}-1}{2}$. Note that $\phi = \alpha + 1$ and $\frac{1}{\phi} = \alpha$. Now, for $i \geq 1$, define

$$f_i = \lfloor (i+1)\alpha \rfloor - \lfloor i\alpha \rfloor \qquad (5)$$

It is known [2] that the infinite word

$$f = f_1 f_2 f_3 \ldots$$

is the word

$$1011010110\ldots$$

This word is the Fibonacci word, with $a = 1$, $b = 0$. We will prove this in a moment. For now, assume it is true and notice that

$$\begin{aligned} v_i = \lfloor (i+1)\phi \rfloor - \lfloor i\phi \rfloor &= \lfloor (i+1)(\alpha+1) \rfloor - \lfloor i(\alpha+1) \rfloor \\ &= \lfloor (i+1)\alpha + (i+1) \rfloor - \lfloor i\alpha + i \rfloor \\ &= \lfloor (i+1)\alpha \rfloor + (i+1) - (\lfloor i\alpha \rfloor + i) \quad \text{since } i \text{ and } i+1 \text{ are integers} \\ &= \lfloor (i+1)\alpha \rfloor - \lfloor i\alpha \rfloor + 1 \\ &= f_i + 1 \end{aligned}$$

Thus, under the assumption that $f$ is the Fibonacci word, we have that

$$v_i = \begin{cases} 2 & \text{if } f_i = 1 \\ 1 & \text{if } f_i = 0 \end{cases}$$

and therefore,

$$y_i = \begin{cases} F_{j+3} & \text{if } f_i = 1 \\ F_{j+2} & \text{if } f_i = 0 \end{cases}.$$

Moreover, it will follow that $v_i$ and $y_i$ also follow the pattern of the Fibonacci word. Therefore, if we can prove that $f$ is the Fibonacci word with 0's and 1's, we will have demonstrated what we want to show.

First, define, as before,

$$g_n = \begin{cases} 1 & \text{if } n = \lfloor k\phi \rfloor \text{ for some integer k} \\ 0 & \text{otherwise} \end{cases}$$

We see that the values of $g_n$ are

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| $g_n$ | 1 | 0 | 1 | 1 | 0 | 1 | 0 | 1 | 1 |
| $k$ | 1 | | 2 | 3 | | 4 | | 5 | 6 |

Let $g$ be the infinite word

$$g = g_1 g_2 g_3 \ldots = 101101011\ldots$$

It seems at first glance that $g = f$. We prove this now. This is lemma 9.1.3 of [2]. We have added in some details.

**Theorem 1.** $g = f$

*Proof.* We observe that $g_n = 1$ if and only if there exists a $k$ such that $n = \lfloor k\phi \rfloor$. This happens exactly when there exists a $k$ such that $n \leq k\phi < n+1$. This happens when there is a $k$ such that $\frac{n}{\phi} \leq k < \frac{n+1}{\phi}$. It follows that $\lfloor \frac{n}{\phi} \rfloor = k-1$ and $\lfloor \frac{n+1}{\phi} \rfloor = k$. To see why, notice that $\frac{1}{\phi} < 1$. Thus, $\frac{n}{\phi}$ and $\frac{n}{\phi} + \frac{1}{\phi}$ differ by a number less than 1. It follows that

$$k - 1 < \tfrac{n}{\phi} < k < \tfrac{n}{\phi} + \tfrac{1}{\phi} < k+1$$

if and only if $g_n = 1$. Therefore, $\lfloor \frac{n}{\phi} \rfloor = k-1$ and $\lfloor \frac{n+1}{\phi} \rfloor = k$. Thus, $\lfloor \frac{n+1}{\phi} \rfloor - \lfloor \frac{n}{\phi} \rfloor = 1$ and since $\alpha = \frac{1}{\phi}$, we have

$$f_n = \lfloor (n+1)\alpha \rfloor - \lfloor n\alpha \rfloor = 1.$$

if and only if $g_n = 1$. All we need to show now is that $f_i$ can only be 0 or 1 for each $i$. Notice

$$\lfloor (i+1)\alpha \rfloor - i\alpha \leq \lfloor (i+1)\alpha \rfloor - \lfloor i\alpha \rfloor \leq (i+1)\alpha - \lfloor i\alpha \rfloor.$$

Also,

$$(i+1)\alpha - \lfloor i\alpha \rfloor = \alpha + i\alpha - \lfloor i\alpha \rfloor < 2, \text{ since } \alpha < 1 \text{ and } i\alpha - \lfloor i\alpha \rfloor < 1$$

and

$$\lfloor (i+1)\alpha \rfloor - i\alpha \geq \lfloor (i+1)\alpha \rfloor - (i+1)\alpha > -1.$$

So

$$-1 < \lfloor (i+1)\alpha \rfloor - i\alpha \leq f_i \leq (i+1)\alpha - \lfloor i\alpha \rfloor < 2.$$

Since $f_i$ is an integer, $f_i = 0$ or 1. Thus, $f_i = 0$ if and only if $g_i = 0$ and $f_i = 1$ if and only if $g_i = 1$. So $g = f$. □

The following theorem, which is lemma 9.1.4 from [2], is all we need to complete the proof of our main theorem.

**Theorem 2.** *$f$ is the Fibonacci word generated by the morphism $\sigma(a) = ab$ and $\sigma(b) = a$, with $a = 1$ and $b = 0$.*

*Proof.* We need to show that $\sigma(f) = f$ where $\sigma$ is the Fibonacci morphism. Since the Fibonacci word is the only word for which this is true, it will follow that $f$ is the Fibonacci word. For $i \geq 1$, define

$$d_i = \sigma(f_i)$$

By the definition of $f_i$,

$$\begin{aligned} d_1 &= \sigma(f_1) = \sigma(1) = 10 \\ d_2 &= \sigma(f_2) = \sigma(0) = 1 \\ d_3 &= \sigma(f_3) = \sigma(1) = 10 \\ &\vdots \end{aligned}$$

Notice that

$$\begin{aligned} \sigma(f) &= \sigma(f_1 f_2 f_3 \ldots) \\ &= \sigma(f_1)\sigma(f_2)\sigma(f_3)\ldots \\ &= d_1 d_2 d_3 \ldots \end{aligned}$$

So, to show that $\sigma(f) = f$, we need to show

$$f = d_1 d_2 d_3 \ldots$$

Now, let $m \geq 1$. Let $n$ be the position of the $m^{th}$ 1 in $\sigma(f)$. Notice that each $d_i$ contains exactly one 1. Thus, this 1 is contained in $d_m$. Consider the word

$$d_1 d_2 \ldots d_{m-1}.$$

The length, or in other words, the number of characters in the word, is given by

$$|d_1 d_2 \ldots d_{m-1}| = \underbrace{(m-1)}_{\text{number of 1's}} + \underbrace{f_1 + f_2 + f_3 + \ldots + f_{m-1}}_{\text{number of 0's}} \tag{6}$$

This is because each $d_i$ has exactly one 1, so there are $m-1$ ones in $d_1 \ldots d_{m-1}$. Also, $d_i = \sigma(f_i)$ is either 1 or 10. Thus, if $f_i = 1$ then $\sigma(f_i) = 10$, and $d_i$ contains one zero. If $f_i = 0$ then $\sigma(f_i) = 1$, and $d_i$ contains no zeros. So $f_1 + f_2 + \ldots + f_{m-1}$ is the number of 0's in $d_1 d_2 \ldots d_{m-1}$. Notice, by definition of $f_i$ (5),

$$\begin{aligned} f_1 + f_2 + \ldots + f_{m-1} &= \lfloor 2\alpha \rfloor - \lfloor \alpha \rfloor + \lfloor 3\alpha \rfloor - \lfloor 2\alpha \rfloor + \lfloor 4\alpha \rfloor - \lfloor 3\alpha \rfloor + \ldots + \lfloor m\alpha \rfloor - \lfloor (m-1)\alpha \rfloor \\ &= \lfloor m\alpha \rfloor - \lfloor \alpha \rfloor \\ &= \lfloor m\alpha \rfloor \end{aligned}$$

The last equality follows since $0 < \alpha < 1$, so $\lfloor \alpha \rfloor = 0$. Now, (6) becomes

$$|d_1 d_2 \ldots d_{m-1}| = (m-1) + \lfloor m\alpha \rfloor.$$

Notice that by the definition of $\sigma$, each $d_i$ begins with 1. So the next element in $d_1 d_2 \ldots d_{m-1}$ is 1. So the $m^{th}$ 1 in $\sigma(f)$ occurs in this position. That is, $n = |d_1 \ldots d_{m-1}| + 1 = m + \lfloor m\alpha \rfloor = \lfloor m + m\alpha \rfloor$ since $m$ is an integer. Then $n = \lfloor m(\alpha+1) \rfloor = \lfloor m\phi \rfloor$. All of this shows that the $n^{th}$ position in $\sigma(f)$ is 1 if and only if there exists an $m$ such that $n = \lfloor m\phi \rfloor$, which occurs if and only if $g_n = 1$. This happens exactly when the $n^{th}$ position of $g$ is 1. So $\sigma(f) = d_1 d_2 \ldots = g$. By Theorem 1, $g = f$, so $\sigma(f) = f$. □

# 4 Further Explorations

In [4], Kimberling attempted to generalize the Wythoff array. He proved that the Wythoff array is the same as the 2nd order Zeckendorf array, another well-known construction. While trying to generalize the Wythoff array, Kimberling also considered the 3rd-order Zeckendorf array:

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 6 | 9 | 13 | 19 | 28 | 41 | 60 | ... |
| 5 | 8 | 12 | 17 | 25 | 37 | 54 | 79 | 116 | 170 | 249 | |
| 7 | 11 | 16 | 23 | 34 | 50 | 73 | 107 | 157 | 230 | 337 | |
| 10 | 15 | 22 | 32 | 47 | 69 | 101 | 148 | 217 | 318 | 466 | |
| 14 | 21 | 31 | 45 | 66 | 97 | 142 | 208 | 306 | 448 | 656 | |
| 18 | 27 | 40 | 58 | 85 | 125 | 183 | 268 | 393 | 576 | 844 | |
| 20 | 30 | 44 | 64 | 94 | 138 | 202 | 296 | 434 | 636 | 932 | |
| 24 | 36 | 53 | 77 | 113 | 166 | 243 | 356 | 522 | 765 | 1121 | |
| 26 | 39 | 57 | 83 | 122 | 279 | 262 | 384 | 563 | 825 | 1209 | |
| ⋮ | | | | | | | | | | | |

The first row is constructed by:

$$
\begin{aligned}
b_0 &= 1 \\
b_1 &= 2 \\
b_2 &= 3 \\
b_n &= b_{n-1} + b_{n-3}
\end{aligned}
$$

The first element in any successive row is the smallest number that does not appear in a previous row. We then write that number as a sum of entries in row one using the largest numbers possible. For example,

$$
\begin{aligned}
5 &= 4 + 1 \\
7 &= 6 + 1 \\
10 &= 9 + 1
\end{aligned}
$$

To generate the rest of the row, we shift the numbers in this sum to the right by one. For example, row two starts with 5, so we see that

$$5 = 4 + 1.$$

So the next entry in row two would be 8 since the sum of the numbers to the right of 4 and 1 is

$$6 + 2 = 8$$

and the third entry in row two would be 12 since the sum of the numbers to the right of 6 and 2 is

$$9 + 3 = 12.$$

Now we look at the column difference sequences:

$$\text{Column 1:} \quad 1 \underbrace{\quad}_{4} 5 \underbrace{\quad}_{2} 7 \underbrace{\quad}_{3} 10 \underbrace{\quad}_{4} 14 \underbrace{\quad}_{4} 18 \underbrace{\quad}_{2} 20 \underbrace{\quad}_{4} 24 \underbrace{\quad}_{2} 26$$

$$\text{Column 2:} \quad 2 \underbrace{\quad}_{6} 8 \underbrace{\quad}_{3} 11 \underbrace{\quad}_{4} 15 \underbrace{\quad}_{6} 21 \underbrace{\quad}_{6} 27 \underbrace{\quad}_{3} 30 \underbrace{\quad}_{6} 36 \underbrace{\quad}_{3} 39$$

$$\text{Column 3:}\quad 3\underbrace{\quad}_{9}12\underbrace{\quad}_{4}16\underbrace{\quad}_{6}22\underbrace{\quad}_{9}31\underbrace{\quad}_{9}40\underbrace{\quad}_{4}44\underbrace{\quad}_{9}53\underbrace{\quad}_{4}57$$

$$\text{Column 4:}\quad 4\underbrace{\quad}_{13}17\underbrace{\quad}_{6}23\underbrace{\quad}_{9}32\underbrace{\quad}_{13}45\underbrace{\quad}_{13}58\underbrace{\quad}_{6}64\underbrace{\quad}_{13}77\underbrace{\quad}_{6}83$$

Now let's compare these difference squences in the columns next to one another:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 4 | 2 | 3 | 4 | 4 | 2 | 4 | 2 |
| 6 | 3 | 4 | 6 | 6 | 3 | 6 | 3 |
| 9 | 4 | 6 | 9 | 9 | 4 | 9 | 4 |
| 13 | 6 | 9 | 13 | 13 | 6 | 13 | 6 |

We notice the pattern:

$$abcaabab...$$

After trial and error, we noticed that this sequence seems to be generated by the morphism

$$\begin{aligned}\sigma(a) &= ab\\ \sigma(b) &= c\\ \sigma(c) &= a\end{aligned}$$

iterated on $a$. We cannot prove this, though, using the methods of this paper which relied heavily on (1). In the conclusion of Kimberling's paper [4], he conjectures that there is likely no formula similar to (1) for generating $m$-order Zeckendorf arrays. This would mean that in order to prove that this morphism generates the column sequences, we would need to use a different approach. It would be interesting also to see if morphisms exist to generate the column difference sequences of the $m$-order Zeckendorf arrays considered by Kimberling.

## References


[1] D.R. Morrison. *A Stolarsky Array of Wythoff Pairs*, A Collection of Manuscripts Related to the Fibonacci Sequence, pp. 134-136. Santa Clara, Calif. The Fibonacci Association, 1980.

[2] Jean-Paul Allouche and Jeffrey Shallit, *Automatic Sequences: Theory, Applications, Generalizations*, Cambridge University Press, Cambridge, 2003.

[3] http://mathworld.wolfram.com/WythoffArray.html

[4] Clark Kimberling, *The Zeckendorf Array Equals the Wythoff Array*, Fibonacci Quarterly, **33**, 3-8, 1995.

[5] Fogg N., *Substitutions in Dynamics, Arithmetics, and Combinatorics*, Lecture Notes in Mathematics 1794, Springer, Berlin, 2002.

[6] Lothaire, M.,*Algebraic Combinatorics on Words*, Encyclopedia of Mathematics and its Applications, Cambridge University Press, Cambridge, 2002.