The Difference Sequences of the Columns of Wythoff’s Array

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1 Introduction

The Fibonacci numbers are defined by $F_0 = 0, F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. For the remainder of this paper, let $\phi = \frac{1+\sqrt{5}}{2}$. The Wythoff array, $W$, was first introduced by David R. Morrison [1] in 1980 as

$$w_{i,1} = \lfloor [i\phi] \phi \rfloor, \quad w_{i,2} = \lfloor [i\phi] \phi^2 \rfloor,$$

where $w_{i,1}$ represents row $i$ column 1 and $w_{i,2}$ represents row $i$ column 2. Recall that the floor of a number $x$ is the largest integer not exceeding $x$. The subsequent entries in each row are then generated by the Fibonacci recurrence.

Example 1. For row 3:

$$\lfloor [3\phi] \phi \rfloor = 6, \quad \lfloor [3\phi] \phi^2 \rfloor = 10$$

So the third row is:

$$6, 10, 16, 26, 42, 68...$$

Morrison [1] proved that the first row of the Wythoff array is the Fibonacci numbers and the first column of the array is computed by the formula

$$w_{i,1} = \lfloor i\phi \rfloor + i - 1,$$

After finding the first number of a row by the above formula, Morrison proved that the rest of $W$ can be computed using the formula

$$w_{i,j+1} = \begin{cases} \lfloor \phi w_{ij} \rfloor + 1 & \text{if } j \text{ is odd} \\ \lfloor \phi w_{ij} \rfloor & \text{if } j \text{ is even} \end{cases} \quad (1)$$

where $w_{ij}$ is row $i$ column $j$ of $W$. Note that the first row of the Wythoff Array is 1, 2, 3, 5, 8, ... which is $F_2, F_3, F_4, F_5, F_6, ...$, so the entry $w_{1,j} = F_{j+1}$. The rest of the Wythoff array is as follows.

<p>| | | | | | | | | | | | |</p>
<table>
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</thead>
</table>
| 1 | 2 | 3 | 5 | 8 | 13| 21| 34| 55| 89| 144| ...
| 4 | 7 | 11| 18| 29| 47| 76 |123|199|322|521|
| 6 |10 |16 |26 |42| 68 |110 |178|288 |466|754|
| 9 |15 |24 |39 |63|102 |165 |267 |432 |699|1131|
|12 |20 |32 |52 |84|136 |220 |356 |576 |932 |1508|
|14 |23 |37 |60 |97|157 |254 |411 |665 |1076 |1741|
|17 |28 |45 |73 |118|191 |309 |500 |809 |1309 |2118|
|19 |31 |50 |81 |131|212 |343 |555 |898 |1453 |2351|
|22 |36 |58 |94 |152|246 |398 |644 |1042 |1686 |2728|


There are a lot of interesting structures in this array [3]. For this paper, we will look at each column and examine the differences between subsequent entries in the columns:

Column 1: 1 3 4 6 9 12 14 17 19 22

Column 2: 2 5 7 10 15 20 23 28 31 36

Column 3: 3 8 11 16 24 32 37 45 50 58

Column 4: 5 13 18 26 39 52 60 73 81 94

Now let’s compare these difference sequences in the columns next to one another:

<p>| | | | | | | | | | |</p>
<table>
<thead>
<tr>
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</tr>
</thead>
<tbody>
<tr>
<td>3</td>
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<td>3</td>
<td>2</td>
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<td>13</td>
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<td>13</td>
<td>8</td>
<td>13</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

By looking at these column differences, we notice a pattern where if the first number is $a$ and the second number is $b$, then the pattern of the column differences is as follows:

$$a \ b \ a \ a \ b \ a \ b \ a \ a \ldots$$

Also, moving to the next row moves $a$ and $b$ to the next Fibonacci numbers. This pattern is actually a very famous pattern known as the Fibonacci word. Dr. Martin Erickson brought this result, without proof, to the attention of my capstone advisor, Dr. David Garth. Dr. Garth then contacted Clark Kimberling[4], perhaps the leading expert on the Wythoff array, about the result. Clark Kimberling responded by saying that he did not recall seeing this result published anywhere. We will later prove that this pattern holds for all columns of the Wythoff array. First we need to discuss the theory of infinite words.

## 2 Monoids, Morphisms, and the Fibonacci Word

Recall that a group is a nonempty set together with a binary operation that is associative and has an identity in which every element in the set has an inverse under the operation. If we remove the requirement that every element has an inverse, then we get a monoid [5].

**Definition 1.** A monoid is a set $M$ together with a binary operation such that the following hold:

1. $x(yz) = (xy)z$ for all $x, y, z \in M$
2. There is a $\lambda \in M$ such that $x\lambda = \lambda x = x$ for all $x \in M$.

**Example 2.** Let $M = \{0,1,2,3,\ldots\}$, the set of nonnegative integers under addition. Then $M$ is a monoid.

**Example 3.** Let $A = \{a,b\}$, and let $A^*$ be the set of finite words over $A$, with $\lambda$ being the empty word.

$$A^* = \{\lambda, a, b, aa, ab, ba, bb, aaa, aab, \ldots\}.$$
So $A^*$ is a monoid, where the operation is concatenation of words. So, for example, if $x = ab$ and $y = baa$ then $xy = abbaa$.

**Definition 2.** A morphism of a monoid $M$ is a map $\sigma : M \to M$ such that $\sigma(xy) = \sigma(x)\sigma(y)$ for all $x, y \in M$.

**Example 4.** The map $\sigma : \mathbb{Z}^+ \cup \{0\} \to \mathbb{Z}^+ \cup \{0\}$ defined by $\sigma(n) = 2n = n + n$ is a morphism of the monoid of nonnegative integers where the operation on integers is addition. This is because

$$\sigma(m + n) = 2(m + n)$$
$$= 2m + 2n$$
$$= \sigma(m) + \sigma(n).$$

**Example 5.** Let $A = \{a, b\}$ and define $\sigma : A^* \to A^*$ by first letting $\sigma(a) = ab$ and $\sigma(b) = a$. We can extend $\sigma$ to a morphism on $A^*$ by concatenation. In other words, if $x, y \in A^*$, and if $\sigma(x)$ and $\sigma(y)$ are defined, then $\sigma(xy)$ is defined to be $\sigma(x)\sigma(y)$. We also define $\sigma(\lambda) = \lambda$. So, for example:

$$\sigma(ab) = \sigma(a)\sigma(b) = aba$$
$$\sigma(ba) = \sigma(b)\sigma(a) = aab$$
$$\sigma(bb) = \sigma(b)\sigma(b) = aa$$
$$\sigma(aa) = \sigma(a)\sigma(a) = abab$$

Continuing in this manner, we extend $\sigma$ to a map on all of $A^*$. Thus $\sigma$ is clearly a morphism. This morphism is known as the Fibonacci morphism [2].

Our next goal is to define the Fibonacci word, which is the word discussed at the end of the introduction. To do this, we use the Fibonacci morphism. First we need to define the notion of an infinite word as a limit of an infinite sequence of finite words.

**Definition 3.** For $A = \{a, b\}$, take a sequence of words $\{u_n\}$, where

$$u_1, u_2, u_3, \ldots \in A^*.$$ 

Then $\{u_n\}$ converges to the infinite word $u$, written $\lim_{n \to \infty} u_n = u$, if and only if every prefix of $u$ is a prefix of all but finitely many $u_n$’s.

**Example 6.** Let:

$$u_1 = a$$
$$u_2 = ab$$
$$u_3 = aba$$
$$u_4 = abab$$
$$u_5 = ababa$$

Clearly under this definition, $u_n$ converges to the infinite word $u = abababab\ldots$. For example, consider the prefix $ababab$ of $u$. Then $ababab$ is a prefix of all $u_n$’s except $u_1, \ldots, u_5$.

We now define the Fibonacci word. From now on, let $\sigma$ be the Fibonacci morphism mentioned earlier. The
idea is to iterate $\sigma$ on $a$. We let $u_n = \sigma^n(a)$, where the exponential notation refers to function composition.

\begin{align*}
u_0 &= \sigma^0(a) = a \\
u_1 &= \sigma^1(a) = ab \\
u_2 &= \sigma^2(a) = \sigma(\sigma(a)) = \sigma(ab) = \sigma(a)\sigma(b) = aba \\
u_3 &= \sigma^3(a) = \sigma(\sigma(\sigma(a))) = \sigma(aba) = \sigma(a)\sigma(b)\sigma(a) = abaab
\end{align*}

Since compositions of morphisms are morphisms, notice that each $u_n$ is a prefix of $u_{n+1}$. To see why, notice that $u_{n+1} = \sigma^{n+1}(a) = \sigma^n(ab) = \sigma^n(a)\sigma^n(b) = u_{n+1}\sigma^n(b)$. So by definition, $\sigma^n(a)$ clearly converges to some infinite word $z$, and we define this word $z$ to be the Fibonacci word. The first few characters in $z$ are as follows,

$z = \lim_{n \to \infty} \sigma^n(a) = abaababaabaababaababa\ldots$

Later we will need to extend the Fibonacci morphism $\sigma$ to infinite words. We do this as follows. If $x = x_1x_2x_3\ldots$ is an infinite word, with $x_i \in \{a, b\}$, then we define $\sigma(x)$ to be the word $\sigma(x_1)\sigma(x_2)\sigma(x_3)\ldots$

We will be using the Fibonacci word to prove our main result about the Wythoff array. However, it is worth mentioning some other interesting properties of the word. Proofs of these properties can be found in [2] and [5]. For the following examples, the Fibonacci word will be denoted as $z$.

**Properties of the Fibonacci Word**

1. The Fibonacci word is a fixed point of $\sigma$ [5]. In other words, $\sigma(z) = z$. This seems plausible, since $\sigma(abaababaab\ldots) = \sigma(a)\sigma(b)\sigma(a)\ldots = abaababaab\ldots$.

Moreover, the Fibonacci word is the only word, finite or infinite, with this property ([5], section 1.2).

2. $\sigma^n(a) = \sigma^{n-1}(a)\sigma^{n-2}(a)$, which tells us that $u_n = u_{n-1}u_{n-2}$ ([2], chapter 7). Each prefix $u_n$ is the concatenation of the previous two words $u_{n-1}$ and $u_{n-2}$. The word can be generated by initial conditions $u_1 = a, u_2 = ab$ as follows:

\begin{align*}
u_1 &= a \\
u_2 &= ab \\
u_3 &= ab a \\
u_4 &= aba ab \\
u_5 &= abaab abab \\
u_6 &= abaababaabaab \\
\vdots
\end{align*}

If we change the initial conditions, we get a different word using the same operation.
3. The length of \( u_n \) is \( F_{n+1} \), the number of \( a \)'s in \( u_n \) is \( F_n \) and the number of \( b \)'s in \( u_n \) is \( F_{n-1} \) ([2], chapter 7).

4. For every \( n \), the Fibonacci word contains \( n + 1 \) subwords of length \( n \) ([5], chapter 6). The following table illustrates this:

<table>
<thead>
<tr>
<th>( n )</th>
<th>Subwords of length ( n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( a, b )</td>
</tr>
<tr>
<td>2</td>
<td>( aa, ab, ba )</td>
</tr>
<tr>
<td>3</td>
<td>( aba, aab, bab, baa )</td>
</tr>
</tbody>
</table>

5. The last two characters of each \( u_n \) alternate between \( ab \) and \( ba \) ([2], chapter 7):

\[
\begin{array}{c}
ab \\
aba \\
abaab \\
abaabaabaabaab \\
abaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaaba
where \( j \geq 1 \) is the index of a fixed column and \( i \geq 1 \). Notice that by (2) we have

\[
\begin{align*}
   w_{i+1,j} - w_{ij} &= \left( \lceil (i+1)\phi \rceil F_{j+1} + iF_j \right) - \left( \lceil i\phi \rceil F_{j+1} + (i-1)F_j \right) \\
   &= \left( \lceil (i+1)\phi \rceil F_{j+1} + iF_j - \lceil i\phi \rceil F_{j+1} - (i-1)F_j \right) \\
   &= \left( \lceil (i+1)\phi \rceil F_{j+1} - \lceil i\phi \rceil F_{j+1} + iF_j - iF_j + F_j \right) \\
   &= \left( \lceil (i+1)\phi \rceil - \lceil i\phi \rceil \right) F_{j+1} + F_j 
\end{align*}
\]

We will show that \( \lceil (i+1)\phi \rceil - \lceil i\phi \rceil \) is always either 1 or 2. It will follow then that either

\[
\begin{align*}
   w_{i+1,j} - w_{ij} &= \left\{ \begin{array}{ll}
      F_{j+1} + F_j = F_{j+2} & \text{if } \lceil (i+1)\phi \rceil - \lceil i\phi \rceil = 1 \\
      2F_{j+1} + F_j = F_{j+1} + F_{j+2} + F_j = F_{j+3} & \text{if } \lceil (i+1)\phi \rceil - \lceil i\phi \rceil = 2
   \end{array} \right.
\end{align*}
\]

Suppose for now that we have shown that \( \lceil (i+1)\phi \rceil - \lceil i\phi \rceil \) is always either 1 or 2. Let us also consider the pattern of the sequence \( \lceil (i+1)\phi \rceil - \lceil i\phi \rceil \). First notice that:

\[
\begin{align*}
   \lfloor \phi \rfloor &= 1 \\
   \lfloor 2\phi \rfloor &= 3 \\
   \lfloor 3\phi \rfloor &= 4 \\
   \lfloor 4\phi \rfloor &= 6 \\
   \lfloor 5\phi \rfloor &= 8 \\
   \lfloor 6\phi \rfloor &= 9 \\
   \lfloor 7\phi \rfloor &= 11 \\
   \lfloor 8\phi \rfloor &= 12 \\
   \lfloor 9\phi \rfloor &= 14 \\
   \lfloor 10\phi \rfloor &= 16 
\end{align*}
\]

Notice also the difference between the consecutive \( \lceil i\phi \rceil \):

\[
\begin{align*}
   1 & 3 & 4 & 6 & 8 & 9 & 11 & 12 & 14 & 16 \\
   2 & 1 & 2 & 2 & 1 & 2 & 2 & 2 & 2 & 2
\end{align*}
\]

Again, we see the pattern of the Fibonacci word, in this case with \( a = 2 \) and \( b = 1 \). We will show that this pattern holds. To summarize where we are going, we will show that

\[
w_{i+1,j} - w_{ij}
\]

is the Fibonacci word, with \( a = F_{j+3}, b = F_{j+2} \), by showing that \( \lceil (i+1)\phi \rceil - \lceil i\phi \rceil \) is the Fibonacci word with \( a = 2, b = 1 \). To do this, we will show that the word \( f_1f_2f_3\ldots \), with \( f_i \) defined by

\[
f_i = \lceil (i+1)\alpha \rceil - \lceil i\alpha \rceil
\]

gives the Fibonacci word with \( a = 1, b = 0 \). Finally, in order to do this, we consider the word \( g_1g_2g_3\ldots \) with \( g_n \) defined by

\[
g_n = \left\{ \begin{array}{ll}
   1 & \text{if } n = \lceil k\phi \rceil \text{ for some integer } k \\
   0 & \text{otherwise}
   \end{array} \right.
\]
and show \( f_n = g_n \) for every \( n \). We will then show that \( \sigma(f) = g = f \), where \( \sigma \) is the Fibonacci morphism. It will follow from the first property of the Fibonacci word mentioned in Section 2 that \( f \) is the Fibonacci word.

Thus, we start by defining
\[
v_i = \lfloor (i + 1) \phi \rfloor - \lfloor i \phi \rfloor\]
and the infinite word
\[
v = v_1v_2v_3v_4v_5... = 21221...
\]
Also, let \( y \) be the infinite word \( y_1y_2y_3... \), where
\[
y_i = w_{i+1,j} - w_{ij}.
\]
Note that by (4),
\[
y_i = \begin{cases} F_{j+3} & \text{if } v_i = 2 \\ F_{j+2} & \text{if } v_i = 1 \end{cases}
\]
Thus, if we can show that \( v \) is the Fibonacci word with \( a = 2 \) and \( b = 1 \), then \( y \) will be the Fibonacci word with \( a = F_{j+3} \) and \( b = F_{j+2} \). This is precisely what we wish to prove. Our strategy will actually involve consideration of the following sequence from chapter 9 of [2]. Let \( \alpha = \frac{\sqrt{5}-1}{2} \). Note that \( \phi = \alpha + 1 \) and \( \frac{1}{\phi} = \alpha \). Now, for \( i \geq 1 \), define
\[
f_i = \lfloor (i + 1) \alpha \rfloor - \lfloor i \alpha \rfloor\]
It is known [2] that the infinite word
\[
f = f_1f_2f_3...
\]
is the word
\[
1011010110...
\]
This word is the Fibonacci word, with \( a = 1 \), \( b = 0 \). We will prove this in a moment. For now, assume it is true and notice that
\[
v_i = \lfloor (i + 1) \phi \rfloor - \lfloor i \phi \rfloor = \lfloor (i + 1)(\alpha + 1) \rfloor - \lfloor i(\alpha + 1) \rfloor
= \lfloor (i + 1)\alpha + (i + 1) \rfloor - \lfloor i\alpha + i \rfloor
= \lfloor (i + 1)\alpha + (i + 1) - (i\alpha + i) \rfloor \text{ since } i \text{ and } i + 1 \text{ are integers}
= \lfloor (i + 1)\alpha \rfloor - \lfloor i\alpha \rfloor + 1
= f_i + 1
\]
Thus, under the assumption that \( f \) is the Fibonacci word, we have that
\[
v_i = \begin{cases} 2 & \text{if } f_i = 1 \\ 1 & \text{if } f_i = 0 \end{cases}
\]
and therefore,
\[
y_i = \begin{cases} F_{j+3} & \text{if } f_i = 1 \\ F_{j+2} & \text{if } f_i = 0 \end{cases}
\]
Moreover, it will follow that \( v_i \) and \( y_i \) also follow the pattern of the Fibonacci word. Therefore, if we can
prove that \( f \) is the Fibonacci word with 0’s and 1’s, we will have demonstrated what we want to show.

First, define, as before,
\[
g_n = \begin{cases} 
1 & \text{if } n = \lfloor k\phi \rfloor \text{ for some integer } k \\
0 & \text{otherwise}
\end{cases}
\]

We see that the values of \( g_n \) are

<table>
<thead>
<tr>
<th>( n )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>( g_n )</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Let \( g \) be the infinite word
\[
g = g_1g_2g_3\ldots = 101101011\ldots
\]

It seems at first glance that \( g = f \). We prove this now. This is lemma 9.1.3 of [2]. We have added in some details.

**Theorem 1.** \( g = f \)

**Proof.** We observe that \( g_n = 1 \) if and only if there exists a \( k \) such that \( n = \lfloor k\phi \rfloor \). This happens exactly when there exists a \( k \) such that \( n \leq k\phi < n + 1 \). This happens when there is a \( k \) such that \( \frac{n}{\phi} \leq k < \frac{n+1}{\phi} \).

It follows that \( \lfloor \frac{n}{\phi} \rfloor = k - 1 \) and \( \lfloor \frac{n+1}{\phi} \rfloor = k \). To see why, notice that \( \frac{1}{\phi} < 1 \). Thus, \( \frac{n}{\phi} \) and \( \frac{n+1}{\phi} \) differ by a number less than 1. It follows that
\[
k - 1 < \frac{n}{\phi} < k < \frac{n+1}{\phi} < k + 1
\]
if and only if \( g_n = 1 \). Therefore, \( \lfloor \frac{n}{\phi} \rfloor = k - 1 \) and \( \lfloor \frac{n+1}{\phi} \rfloor = k \). Thus, \( \lfloor \frac{n+1}{\phi} \rfloor - \lfloor \frac{n}{\phi} \rfloor = 1 \) and since \( \alpha = \frac{1}{\phi} \), we have
\[
f_n = \lfloor (n+1)\alpha \rfloor - \lfloor n\alpha \rfloor = 1.
\]
if and only if \( g_n = 1 \). All we need to show now is that \( f_i \) can only be 0 or 1 for each \( i \). Notice
\[
\lfloor (i+1)\alpha \rfloor - i\alpha \leq \lfloor (i+1)\alpha \rfloor - \lfloor i\alpha \rfloor \leq (i+1)\alpha - i\alpha.
\]

Also,
\[
(i+1)\alpha - \lfloor i\alpha \rfloor = \alpha + i\alpha - \lfloor i\alpha \rfloor < 2, \text{ since } \alpha < 1 \text{ and } i\alpha - \lfloor i\alpha \rfloor < 1
\]
and
\[
\lfloor (i+1)\alpha \rfloor - i\alpha \geq \lfloor (i+1)\alpha \rfloor - (i+1)\alpha > -1.
\]
So
\[
-1 < \lfloor (i+1)\alpha \rfloor - i\alpha \leq f_i \leq (i+1)\alpha - \lfloor i\alpha \rfloor < 2.
\]
Since \( f_i \) is an integer, \( f_i = 0 \) or 1. Thus, \( f_i = 0 \) if and only if \( g_i = 0 \) and \( f_i = 1 \) if and only if \( g_i = 1 \). So \( g = f \).

The following theorem, which is lemma 9.1.4 from [2], is all we need to complete the proof of our main theorem.

**Theorem 2.** \( f \) is the Fibonacci word generated by the morphism \( \sigma(a) = ab \) and \( \sigma(b) = a \), with \( a = 1 \) and \( b = 0 \).
Proof. We need to show that $\sigma(f) = f$ where $\sigma$ is the Fibonacci morphism. Since the Fibonacci word is the only word for which this is true, it will follow that $f$ is the Fibonacci word. For $i \geq 1$, define

$$d_i = \sigma(f_i)$$

By the definition of $f_i$,

$$d_1 = \sigma(f_1) = \sigma(1) = 10$$
$$d_2 = \sigma(f_2) = \sigma(0) = 1$$
$$d_3 = \sigma(f_3) = \sigma(1) = 10$$
$$\vdots$$

Notice that

$$\sigma(f) = \sigma(f_1f_2f_3\ldots) = \sigma(f_1)\sigma(f_2)\sigma(f_3)\ldots = d_1d_2d_3\ldots$$

So, to show that $\sigma(f) = f$, we need to show

$$f = d_1d_2d_3\ldots$$

Now, let $m \geq 1$. Let $n$ be the position of the $m^{th}$ 1 in $\sigma(f)$. Notice that each $d_i$ contains exactly one 1. Thus, this 1 is contained in $d_m$. Consider the word

$$d_1d_2\ldots d_{m-1}.$$ 

The length, or in other words, the number of characters in the word, is given by

$$|d_1d_2\ldots d_{m-1}| = (m-1) + (f_1 + f_2 + f_3 + \ldots + f_{m-1})$$

which occurs if and only if $g_n = 1$. This happens exactly when the $n^{th}$ position of $g$ is 1. So $\sigma(f) = d_1d_2\ldots = g$. By Theorem 1, $g = f$, so $\sigma(f) = f$. 

\[\square\]
4 Further Explorations

In [4], Kimberling attempted to generalize the Wythoff array. He proved that the Wythoff array is the same as the 2nd order Zeckendorf array, another well-known construction. While trying to generalize the Wythoff array, Kimberling also considered the 3rd-order Zeckendorf array:

\[
\begin{align*}
1 & \quad 2 & \quad 3 & \quad 4 & \quad 6 & \quad 9 & \quad 13 & \quad 19 & \quad 28 & \quad 41 & \quad 60 & \ldots \\
5 & \quad 8 & \quad 12 & \quad 17 & \quad 25 & \quad 37 & \quad 54 & \quad 79 & \quad 116 & \quad 170 & \quad 249 \\
7 & \quad 11 & \quad 16 & \quad 23 & \quad 34 & \quad 50 & \quad 73 & \quad 107 & \quad 157 & \quad 230 & \quad 337 \\
10 & \quad 15 & \quad 22 & \quad 32 & \quad 47 & \quad 69 & \quad 101 & \quad 148 & \quad 217 & \quad 318 & \quad 466 \\
14 & \quad 21 & \quad 31 & \quad 45 & \quad 66 & \quad 97 & \quad 142 & \quad 208 & \quad 306 & \quad 448 & \quad 656 \\
18 & \quad 27 & \quad 40 & \quad 58 & \quad 85 & \quad 125 & \quad 183 & \quad 268 & \quad 393 & \quad 576 & \quad 844 \\
20 & \quad 30 & \quad 44 & \quad 64 & \quad 94 & \quad 138 & \quad 202 & \quad 296 & \quad 434 & \quad 636 & \quad 932 \\
24 & \quad 36 & \quad 53 & \quad 77 & \quad 113 & \quad 166 & \quad 243 & \quad 356 & \quad 522 & \quad 765 & \quad 1121 \\
26 & \quad 39 & \quad 57 & \quad 83 & \quad 122 & \quad 179 & \quad 262 & \quad 384 & \quad 563 & \quad 825 & \quad 1209 \\
\vdots
\end{align*}
\]

The first row is constructed by:

\[
\begin{align*}
b_0 &= 1 \\
b_1 &= 2 \\
b_2 &= 3 \\
b_n &= b_{n-1} + b_{n-3}
\end{align*}
\]

The first element in any successive row is the smallest number that does not appear in a previous row. We then write that number as a sum of entries in row one using the largest numbers possible. For example,

\[
\begin{align*}
5 &= 4 + 1 \\
7 &= 6 + 1 \\
10 &= 9 + 1
\end{align*}
\]

To generate the rest of the row, we shift the numbers in this sum to the right by one. For example, row two starts with 5, so we see that

\[
5 = 4 + 1.
\]

So the next entry in row two would be 8 since the sum of the numbers to the right of 4 and 1 is

\[
6 + 2 = 8
\]

and the third entry in row two would be 12 since the sum of the numbers to the right of 6 and 2 is

\[
9 + 3 = 12.
\]

Now we look at the column difference sequences:

Column 1: 1 5 7 10 14 18 20 24 26

\[
\begin{align*}
4 & \quad 2 & \quad 3 & \quad 4 & \quad 4 & \quad 4 & \quad 2 & \quad 4 & \quad 2
\end{align*}
\]

Column 2: 2 8 11 15 21 27 30 36 39

\[
\begin{align*}
6 & \quad 3 & \quad 4 & \quad 6 & \quad 6 & \quad 3 & \quad 6 & \quad 3
\end{align*}
\]
Now let’s compare these difference sequences in the columns next to one another:

\[
\begin{array}{cccccccc}
4 & 2 & 3 & 4 & 4 & 2 & 4 & 2 \\
6 & 3 & 4 & 6 & 6 & 3 & 6 & 3 \\
9 & 4 & 6 & 9 & 9 & 4 & 9 & 4 \\
13 & 6 & 9 & 13 & 13 & 6 & 13 & 6 \\
\end{array}
\]

We notice the pattern:

\[abcaabab…\]

After trial and error, we noticed that this sequence seems to be generated by the morphism

\[
\begin{align*}
\sigma(a) &= ab \\
\sigma(b) &= c \\
\sigma(c) &= a
\end{align*}
\]

iterated on \(a\). We cannot prove this, though, using the methods of this paper which relied heavily on (1). In the conclusion of Kimberling’s paper [4], he conjectures that there is likely no formula similar to (1) for generating \(m\)-order Zeckendorf arrays. This would mean that in order to prove that this morphism generates the column sequences, we would need to use a different approach. It would be interesting also to see if morphisms exist to generate the column difference sequences of the \(m\)-order Zeckendorf arrays considered by Kimberling.

References


