A Graphical Analysis of Midy's Theorem

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1 Introduction

Consider the fraction $\frac{1}{7}$. The decimal expansion of this fraction is $.\overline{142857}$. At first glance, this may seem to be an ordinary decimal, but on closer examination, some interesting characteristics emerge. First, note that there is an even number of repeating digits. Then split the group of repeating digits into two "blocks": 142 and 857. Then if we add these two blocks together we get 999. This property, formalized as Midy's Theorem,was utilized by Carl Friedrich Hindenburg in 1776 [1]. This theorem, which we will examine in this paper, became known as Midy's Theorem due to a pamphlet published by E. Midy in Nantes, France, 1836. This theorem has been more fully developed and expanded upon in numerous papers and articles [1],[4]. The characteristic of blocks of repeating digits summing to a string of nines hold for more than just 2 blocks (Midy's Theorem). For example, $14 + 28 + 57 = 99$ demonstrates that the 3 block property also holds for $\frac{1}{7}$. These block properties also hold for more than just prime reciprocals, as discussed in [1]. In this paper, we seek to prove Midy's Theorem, and develop a graphical representation of fractions of repeating decimals, from which we can give a necessary and sufficient condition under which the conclusion of Midy's Theorem holds. From this condition, we have another proof of Midy's Theorem.

2 Representations of Numbers in Any Base

Necessary for our discussion is the ability to uniquely write any number in any base. While we primarily focus on base 10 in this paper, note that this theorem holds for any base. This theorem can be found as Exercise 2.14 in [3].

Theorem 1. *Let b be an integer greater than 1. Then every positive integer n can be written uniquely in the form*

$$n = d_k b^k + d_{k-1} b^{k-1} + ... + d_1 b + d_0 \tag{1}$$

where with $k \geq 0$, each d_j is an integer such that $0 \leq d_j \leq b-1$, and $d_k \neq 0$

Proof. Let $n > 1$. Then let $k \geq 0$ be such that $b^k \leq n < b^{k+1}$. By repeatedly applying the division algorithm, we obtain a sequence of quotient and remainder pairs such that

$$\begin{array}{rcll} n & = & q_k b^k + r_k & 0 \leq r_k < b^k \\ r_k & = & q_{k-1} b^{k-1} + r_{k-1} & 0 \leq r_{k-1} < b^{k-1} \\ r_{k-1} & = & q_{k-2} b^{k-2} + r_{k-2} & 0 \leq r_{k-2} < b^{k-2} \\ & \vdots & & \\ r_2 & = & q_1 b^1 + r_1 & 0 \leq r_1 < b \end{array}$$

Then

$$
\begin{aligned}
n = q_k b^k + r_k \quad &= q_k b^k + q_{k-1} b^{k-1} + r_{k-1} \\
&= q_k b^k + q_{k-1} b^{k-1} + q_{k-2} b^{k-2} + r_{k-2} \\
&\vdots \\
&= q_k b^k + q_{k-1} b^{k-1} + ... + q_2 b^2 + q_1 b + r_1
\end{aligned}
$$

Let $d_k = q_k,\ d_{k-1} = q_{k-1},\ ...\ d_1 = q_1,\ d_0 = r_1$. Since $b^k \leq n$, it follows that $d_k \neq 0$. Also, since $d_k \geq b$ implies $n = q_k b^k + r_k \geq q_k b^k \geq b^{k+1}$, which is a contradiction, we know $d_k < b$. Similarly, we can show that $d_i < b$ for $0 \leq i \leq k-1$. □

As an example, consider the number 23. We can write $23 = 1 \cdot 2^4 + 0 \cdots 2^3 + 1 \cdot 2^2 + 1 \cdot 2 + 1$, demonstrating 10111 is the base 2 representation of 23. Note that all real numbers can be expressed in base b decimal representation. For example, the repeating decimal $.\overline{01}$ in base 2 is

$$0 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2^2} + 0 \cdot \frac{1}{2^3} + \cdots = \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \cdots \tag{2}$$

This is just the geometric series whose sum is $\frac{4}{3}$ in base 10.

For notation, we will use $[d_k d_{k-1} \cdots d_0]$ to represent (1).

3 Proving Midy's Theorem

We shall prove Midy's Theorem by emulating the proof given in Ginsberg's article [4]. Ginsberg begins by stating a needed fact about repeating decimals, which we will also prove. In the rest of this paper, we shall assume that any fraction stated in any of the subsequent theorems will satisfy the hypothesis of this theorem.

Theorem 2. *Any reduced fraction with prime denominator $p > 5$ is purely periodic. Such a number $\frac{m}{p}$ where $m < p$ can be represented as*

$$\frac{m}{p} = .\overline{a_1 \ldots a_{\lambda-1} a_\lambda} = \frac{[a_1 \ldots a_{\lambda-1} a_\lambda]}{10^\lambda - 1}. \tag{3}$$

Proof. First, we will prove the first part of this theorem. Let $\frac{m}{p}$ be a fraction where $p > 5$. Let us consider using long division to divide m by p. Then we would have:

$$
\begin{array}{l}
\quad .a_1 a_2 a_3 ... a_\lambda a_1 ... \\
p \overline{\smash{)} m.\ 0\ 0\ 0\ 0\ 0\ 0} \\
\quad \underline{a_1 \cdot p} \\
\quad r_1 0 \\
\quad\quad \underline{a_2 \cdot p} \\
\quad\quad r_2 0 \\
\quad\quad\quad \underline{a_3 \cdot p} \\
\quad\quad\quad r_3 0 \\
\quad\quad\quad\quad \vdots \\
\quad\quad\quad\quad r_\lambda 0 \\
\quad\quad\quad\quad \underline{a_\lambda \cdot p} \\
\quad\quad\quad\quad\quad \vdots
\end{array}
$$

Thus we see this important result in base 10 which we will use later on as well:

$$(10^{\ell}m) \bmod p = \ell^{th}\text{remainder}. \tag{4}$$

Then we know that the decimal representation for $\frac{m}{p}$ is purely periodic if and only if there exists an $\ell \geq 2$ such that $10^{\ell}m \equiv m \bmod p$ since this will begin the cycle of repeating digits again, starting with the first digit after the decimal point, giving ℓ as our period. Note that when $\ell = 1$, we have

$$10m \equiv m \bmod p \tag{5}$$

which implies that

$$10 \equiv 1 \bmod p \tag{6}$$

This would imply that $p|9$, so $p = 3$, but we assumed that $p > 5$. Then from Euler's Theorem[3], we have that there exists an ℓ such that $10^{\ell} \equiv 1 \bmod p$ since $gcd(10, p) = 1$. Thus we see for a purely periodic fraction $\frac{m}{p}$, since $10^{\ell} \equiv 1 \bmod p$, the period is the order of $10 \bmod p$. Note that we did not need a base of ten in the result. The fraction $\frac{m}{p}$ will be periodic as long as $gcd(b, p) = 1$. The period will be the order of $b \bmod p$.

Now let us prove the second part of this theorem. Let $\frac{m}{p}$ be the decimal representation

$$\frac{m}{p} = \overline{.a_1 a_2 \cdots a_\lambda} = .a_1 a_2 \cdots a_\lambda a_1 \cdots a_\lambda \cdots$$

Then we can represent $\frac{m}{p}$ as

$$\begin{aligned}
\frac{m}{p} &= \frac{a_1}{10} + \frac{a_2}{10^2} + \frac{a_3}{10^3} + \cdots + \frac{a_\lambda}{10^\lambda} + \frac{a_1}{10^{\lambda+1}} + \cdots \\
&= \left(\frac{a_1}{10} + ... + \frac{a_\lambda}{10^\lambda}\right) + \frac{1}{10^\lambda}\left(\frac{a_1}{10} + ... + \frac{a_\lambda}{10^\lambda}\right) + \frac{1}{10^{2\lambda}}\left(\frac{a_1}{10} + ... + \frac{a_\lambda}{10^\lambda}\right) + ... \\
&= \left(\frac{a_1}{10} + ... + \frac{a_\lambda}{10^\lambda}\right)\left[1 + \frac{1}{10^\lambda} + \frac{1}{10^{2\lambda}} + ...\right] \\
&= \left(\frac{a_1}{10} + ... + \frac{a_\lambda}{10^\lambda}\right)\sum_{i=0}^{\infty}\left(\frac{1}{10^\lambda}\right)^i \\
&= \left(\frac{a_1}{10} + ... + \frac{a_\lambda}{10^\lambda}\right)\left(\frac{1}{1 - \frac{1}{10^\lambda}}\right) \\
&= \left(\frac{a_1}{10} + ... + \frac{a_\lambda}{10^\lambda}\right)\left(\frac{10^\lambda}{10^\lambda - 1}\right) \\
&= \left(\frac{a_1}{10} + ... + \frac{a_\lambda}{10^\lambda}\right)\left(\frac{10^\lambda}{10^\lambda - 1}\right) \\
&= \left(\frac{10^{\lambda-1}a_1 + 10^{\lambda-2}a_2 + \cdots + a_\lambda}{10^\lambda}\right)\left(\frac{10^\lambda}{10^\lambda - 1}\right) \\
&= \frac{[a_\lambda a_{\lambda-1} \ldots a_1]}{10^\lambda - 1}
\end{aligned} \tag{7}$$

□

We now prove Midy's Theorem.[4]

Theorem 3. *Midy's Theorem: Let p be a prime such that $p > a$, and let m be a natural number such that $m < p$. Suppose $\frac{m}{p}$ has a period $\lambda = 2k$. Let $\frac{m}{p}$ have decimal representation*

$$\frac{m}{p} = \overline{.a_1 a_2 \cdots a_\lambda},$$

and let A be $[a_1 a_2 \cdots a_k]$ and B be $[a_{k+1} a_{k+2} \cdots a_\lambda]$. Then

$$A + B = 10^k - 1. \tag{8}$$

We say that any $\frac{m}{p}$ that satisfies Midy's Theorem has the 2 block property.

Proof. Since we assume $\frac{m}{p}$ has period $2k$, by Theorem 1, we can write its decimal expansion as

$$\frac{m}{p} = \frac{\left(10^k A + B\right)}{\left(10^{2k} - 1\right)} \tag{9}$$

For example, if $\frac{m}{p} = \frac{1}{7}$, then $A = 142$ and $B = 857$. We can see that $\lambda = 6$ and $k = 3$. Then

$$\frac{1}{7} = \frac{10^3(142) + (857)}{10^6 - 1} = \frac{142,857}{999,999} \tag{10}$$

Then using difference of squares, we see that

$$\frac{m}{p} = \frac{10^k A + B}{10^{2k} - 1} = \frac{10^k A + B}{\left(10^k - 1\right)\left(10^k + 1\right)}. \tag{11}$$

Thus by cross multiplying, we have

$$\left(10^k A + B\right) p = m\left(10^k - 1\right)\left(10^k + 1\right). \tag{12}$$

Since p is prime, by Euclid's Lemma, p must divide at least one of m, $(10^k - 1)$ and $(10^k + 1)$. We know that p does not divide m by assumption, so p must divide either $(10^k - 1)$ or $(10^k + 1)$. If $p|(10^k - 1)$, then $10^k \equiv 1 \bmod p$. This shows that the period of $\frac{m}{p}$ must divide k, since the period is the order of 10 mod p. However,this is a contradiction by our hypothesis since the period is $2k$. Thus p must divide $10^k + 1$. Rearranging, we have

$$\begin{aligned} \frac{m(10^k + 1)}{p} &= \frac{10^k A + B}{10^k - 1} \\ &= \frac{10^k A - A + A + B}{10^k - 1} \\ &= A + \frac{A + B}{10^k - 1}. \end{aligned} \tag{13}$$

Since $p|(10^k + 1)$, we know the left side of the equation is a whole number and thus the right side is as well. Then since A is a whole number, $\frac{A+B}{10^k - 1}$ must be as well. Now note that $(10^k - 1)$ is the largest k digit whole number. Since A is k digits and B is k digits, we have

$$A \leq (10^k - 1) \tag{14}$$

and

$$B \leq (10^k - 1). \tag{15}$$

Thus

$$A + B \leq 2(10^k - 1). \tag{16}$$

Now to show a contradiction, suppose $A = B = (10^k - 1)$. Then

$$\begin{aligned} 10^k A + B &= 10^k(10^k - 1) + (10^k - 1) \\ &= 10^{2k} - 1 \\ &= \underbrace{999 \cdots 999}_{2k \text{ 9s}} \end{aligned} \tag{17}$$

However, this is a contradiction since this would imply

$$\frac{m}{p} = \frac{10^{2k} - 1}{10^{2k} - 1} = 1. \tag{18}$$

Thus

$$A + B < 2(10^k - 1). \tag{19}$$

Since we have shown that $\frac{A+B}{10^k-1}$ is a whole number, then we have our desired result

$$A + B = 10^k - 1. \tag{20}$$

Notice that Midy's Theorem holds in an arbitrary base b as long as $gcd(b, p) = 1$. □

4 A Graphical Representation of Midy's Theorem

By defining a function, as in [2], we can graphically analyze repeating decimals.

Definition 1. Let a, b, and $n > 3$ be positive integers with $(n, a) = 1$ and $b > 1$. If r_i is the remainder produced at step i of the base b long division of $\frac{a}{n}$, the remainder produced at the$(i + 1)$st step is given by

$$r_{i+1} = F_{b:n}(r_i) = br_i(\text{mod} n). \tag{21}$$

For example, let us consider the remainders from long division in base ten for $\frac{1}{7}$.

$$
\begin{array}{l}
\quad\;\, .142857 \\
7\,\overline{)\mathbf{1}.000000} \\
\quad\;\; \underline{7} \\
\quad\;\; \mathbf{3}0 \\
\quad\;\; \underline{28} \\
\quad\;\;\; \mathbf{2}0 \\
\quad\;\;\; \underline{14} \\
\quad\;\;\;\; \mathbf{6}0 \\
\quad\;\;\;\; \underline{56} \\
\quad\;\;\;\;\; \mathbf{4}0 \\
\quad\;\;\;\;\; \underline{35} \\
\quad\;\;\;\;\;\; \mathbf{5}0 \\
\quad\;\;\;\;\;\; \underline{49} \\
\quad\;\;\;\;\;\;\; \mathbf{1}
\end{array}
$$

Then starting with $a = r_0 : 1 \to 3 \to 2 \to 6 \to 4 \to 5 \to 1 \to \ldots$ Thus given r_0 we can determine all other remainders using our remainder function:

$$r_1 = F_{10:7}(r_0) = 10r_0(\text{mod}7) = 10(1)(\text{mod}7) = 3$$

$$r_2 = F_{10:7}(r_1) = 10r_1(\text{mod}7) = 10(3)(\text{mod}7) = 2$$
$$r_3 = F_{10:7}(r_2) = 10r_2(\text{mod}7) = 10(2)(\text{mod}7) = 6$$
$$r_4 = F_{10:7}(r_3) = 10r_3(\text{mod}7) = 10(6)(\text{mod}7) = 4$$
$$r_5 = F_{10:7}(r_4) = 10r_4(\text{mod}7) = 10(4)(\text{mod}7) = 5$$
$$r_6 = F_{10:7}(r_5) = 10r_5(\text{mod}7) = 10(5)(\text{mod}7) = 1$$

As $r_6 = r_0 = 1$, this sequence will repeat.

Now we can create a graphical representation of the sequence of remainders for various fractions in any base using this remainder function. We will mainly be concerned with base 10 representations. To do so, we proceed through the following steps [2]:

1. Let r_0 be a in our fraction $\frac{a}{n}$.
2. Beginning with $i = 0$, draw a vertical line from (r_i, r_i) to the point $(r_i, F_{b:n}(r_i)) = (r_i, r_{i+1})$
3. From there, draw a horizontal line to $(F(r_i), F(r_i)) = (r_{i+1}, r_{i+1})$
4. Increase i by 1 iteratively and repeat the procedure.

When considering fractions in general, if the remainder becomes zero at some r_i, then we stop the process as r_{i-1}. However, we know this will not happen for purely periodic fractions. The graph of the remainder function created by these steps for some fraction $\frac{a}{n}$ is what we will refer to as the **graph of the fraction**. From our steps we can also see that the sequence of the remainders determines the graph of the fraction.

Figure 1 is an example of the graph of $\frac{1}{7}$ in base 10 produced by following these steps. The points are $(1,1) \to (1,3) \to (3,3) \to (3,2) \to \ldots \to (5,1) \to (1,1)$. Figures 2-5 show some graphs of some other fractions and are listed at the end of the paper. Now if we look at

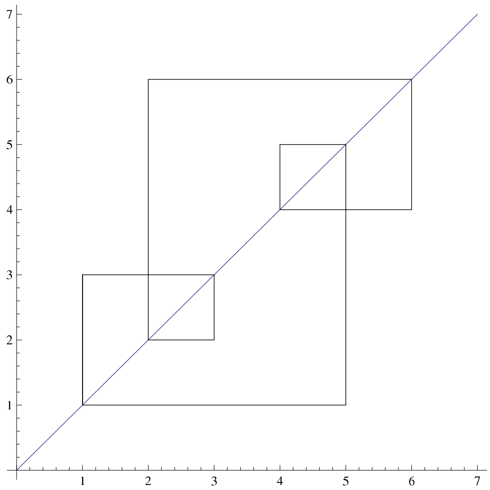

Figure 1: The graph of $\frac{1}{7}$ base 10

Figures 1, 2, and 3, we will see a similar characteristic, in contrast to Figures 4 and 5. This characteristic is what we call rotational symmetry, and we will define it as follows:

Definition 2. A graph of some fraction with denominator n in base b has rotational symmetry if, by rotating the graph 180° about $(\frac{n}{2}, \frac{n}{2})$, we get the same graph.

Figures 1, 2, and 3 all have rotational symmetry while Figures 4 and 5 do not.

Theorem 4. *If $gcd(n,a) = 1$ and n is odd, with $a < n$ and $gcd(b,n) = 1$, then there exists some natural number m, $0 < m < n$, such that for each natural number i we have $r_i + r_{i+m} = n$ if and only if the graph of $\frac{a}{n}$ in base b has rotational symmetry.[2]*

Proof. We will first prove the forward conditional. Since we know there exists some m, $0 < m < n$, such that for each natural number i we have

$$r_i + r_{i+m} = n, \tag{22}$$

let s be the smallest integer for which that is true. Then letting $i = s$, we have

$$r_s + r_{2s} = n, \tag{23}$$

and letting $i = 0$, we have

$$r_0 + r_s = n. \tag{24}$$

By combining these two equations, we see that $r_0 = r_{2s}$. Thus the length of the sequence of remainders before it begins repeating is $2s$. Furthermore, the sequence of remainders can be grouped into two halves, $r_0, r_1, \ldots r_{s-1}$ and$r_s, r_{s+1}, \ldots r_{2s-1}$, where $r_0 + r_s = n$,

$r_1 + r_{s+1} = n$,...and $r_{s-1} + r_{2s-1} = n$. Since $r_i + r_{s+i} = n$ for $0 \le i \le s-1$, either $r_i < \frac{n}{2}$ and $r_{s+i} > \frac{n}{2}$,or vice versa. Assume the former without loss of generality. Then

$$r_i + r_{s+i} = \frac{n}{2} + \frac{n}{2}, \tag{25}$$

implying that

$$r_{s+i} - \frac{n}{2} = \frac{n}{2} - r_i. \tag{26}$$

Thus, (r_i, r_i) and (r_{s+i}, r_{s+i}) are equidistant from $(\frac{n}{2}, \frac{n}{2})$, so (r_i, r_i) rotates to (r_{s+i}, r_{s+i}) for all i. Since (r_{i+1}, r_{i+1}) rotates to (r_{s+i+1}, r_{s+i+1}) and (r_i, r_i) rotates to (r_{s+i}, r_{s+i}), it follows that (r_i, r_{i+1}) rotates to (r_{s+i}, r_{s+i+1}) .Thus when rotated 180° about $(\frac{n}{2}, \frac{n}{2})$, each remainder will be rotated to its counterpart in the other half, and we see that the whole graph is rotationally symmetric.

Now let us prove the converse of our biconditional. Suppose the graph of $\frac{a}{n}$ base b, with $r_0 = a$, has rotational symmetry. From our definition of rotational symmetry, we know it rotates around $(\frac{n}{2}, \frac{n}{2})$. Assume (r_0, r_0) rotates to some other point (r_s, r_s). Then we know (r_0, r_1) rotates to (r_s, r_{s+1}). Then (r_1, r_1) rotates to (r_{s+1}, r_{s+1}). Continuing this process, we will eventually get to back to (r_0, r_0). Then since every rotational pair is equidistant from $(\frac{n}{2}, \frac{n}{2})$, for any i, we have

$$\left|\frac{n}{2} - r_i\right| = \left|\frac{n}{2} - r_{s+i}\right|. \tag{27}$$

Thus we see that $r_i + r_{s+i} = n$ since by the nature of rotation,since (r_i, r_i) rotates to (r_{s+i}, r_{s+i}), one remainder is less than $\frac{n}{2}$ and one is greater. Thus we know there exists some number m, $0 < m < n$, such that for each natural number i, we have $r_i + r_{i+m} = n$. □

Now let us show that the remainders of a fraction $\frac{a}{p}$, where p is prime and $gcd(p, b) = 1$, that has the 2-block property will imply the first part of Theorem 4.

Theorem 5. *Let $\frac{a}{p}$ be a fraction with even period, where p is prime, $gcd(p, b) = 1$, and $a < p$. Then $\frac{a}{p}$ satisfies the 2-block property if and only if $r_i + r_{i+s} = p$ for every i.*

Proof. Let us prove the forward direction first. Let $l = 2s$ be the period of $\frac{a}{p}$. Then since l is the order of b mod p, we know l is the smallest number such that $b^l \equiv 1 \bmod p$. Thus p must divide $b^l - 1$. Since

$$b^l = \left(b^{\frac{l}{2}} - 1\right)\left(b^{\frac{l}{2}} + 1\right), \tag{28}$$

then by Euclid's Lemma, we know p must divide either $\left(b^{\frac{l}{2}} - 1\right)$ or $\left(b^{\frac{l}{2}} + 1\right)$. Suppose p divides $\left(b^{\frac{l}{2}} - 1\right)$. Then $b^{\frac{l}{2}}$ is congruent to 1 mod p, but this would be a contradiction since $\frac{l}{2}$ is smaller than l. Thus, p must divide $\left(b^{\frac{l}{2}} + 1\right)$. Thus, $b^{\frac{l}{2}}$ is congruent to-1 mod p. Since

$$b^{\frac{l}{2}} \equiv -1 \bmod p, \tag{29}$$

we have

$$b^{\frac{l}{2}}10^i \equiv -b^i \bmod p \tag{30}$$

by mutliplying by b^i. Then

$$b^{\frac{l}{2}+i} \equiv -b^i \bmod p, \tag{31}$$

which gives us

$$b^{\frac{l}{2}+i} + b^i \equiv 0 \bmod p. \tag{32}$$

Since $b^{\frac{l}{2}+i} \bmod p$ is just $r_{\frac{l}{2}+i}$ and $b^i \bmod p = r_i$, then we have

$$r_{\frac{l}{2}+i} + r_i \equiv 0 \bmod p. \tag{33}$$

Thus p divides $(r_{\frac{l}{2}+i} + r_i)$, but the sum of any two remainders must be less than or equal to p. Thus

$$r_{\frac{l}{2}+i} + r_i = p. \tag{34}$$

Now supppose there exist some natural number m, $0 < m < p$, such that for each natural number i, we have $r_i + r_{i+m} = p$ for some fraction $\frac{a}{p}$ where $gcd(p, a) = 1$ and $a < p$. To show that $\frac{a}{p}$ satifies 3 (Midy's Theorem), we only need to show

$$b - 1 = a_i + a_{i+m} \tag{35}$$

where b is our base and the a_i's are the digits in our decimal representation of $\frac{a}{p}$[2]. Then using the long division algorithm again, we know that for each i,

$$br_{i-1} = pa_i + r_i \tag{36}$$

By the same method, we know

$$br_{i-1+m} = pa_{i+m} + r_{i+m} \tag{37}$$

Then by adding these two equations together, we get

$$b(r_{i-1} + r_{i-1+m}) = p(a_i + a_{i+m}) + (r_i + r_{i+m}) \tag{38}$$

We know by assumption that $r_{i-1} + r_{i-1+m} = p$ and $r_i + r_{i+m} = p$, so we have

$$bp = p(a_i + a_{i+m}) + p. \tag{39}$$

Then canceling out the p's, we have

$$b = (a_i + a_{i+m}) + 1, \tag{40}$$

which implies that

$$b - 1 = a_i + a_{i+m}. \tag{41}$$

Thus we have our desired result. □

Our final theorem follows directly from Theorems 4 and 5.

Theorem 6. *A fraction $\frac{m}{p}$ satisfies the 2-block property if and only if the graph of the fraction's remainder function is rotationally symmetric.*

References

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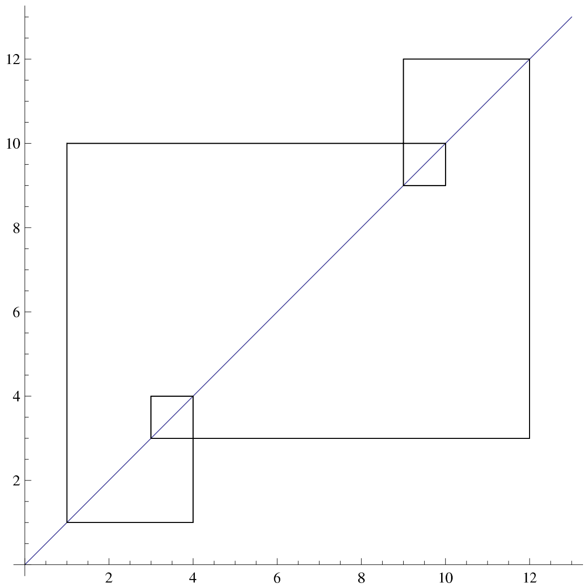

Figure 2: The graph of $\frac{1}{13}$ base 10

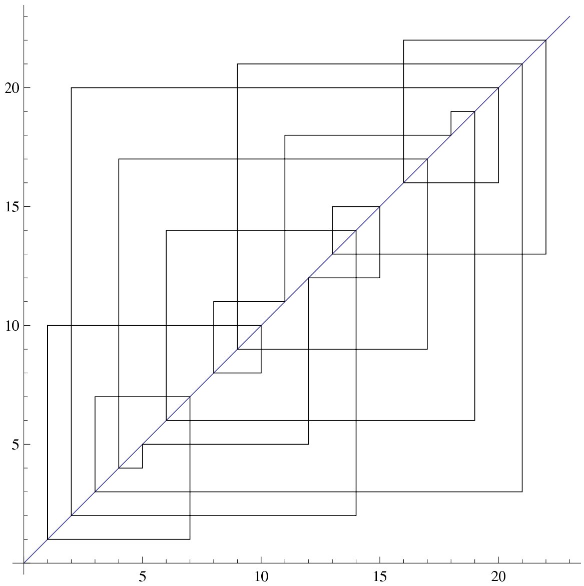

Figure 3: The graph of $\frac{1}{23}$ base 10

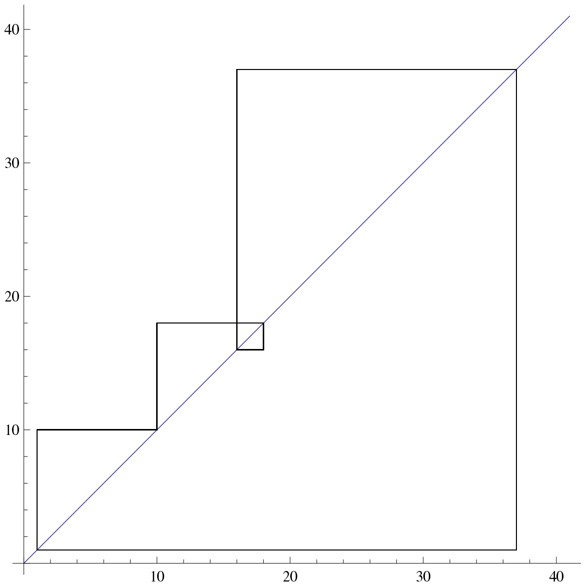

Figure 4: The graph of $\frac{1}{41}$ base 10

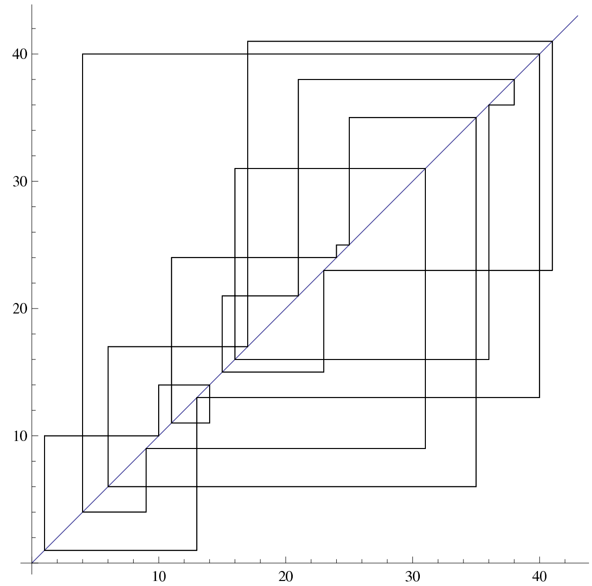

Figure 5: The graph of $\frac{1}{43}$ base 10