A Curious Case of a Continued Fraction

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1 Introduction

The primary purpose of this paper is to combine the studies of continued fractions, the Fibonacci sequence, base two representation, and Fibonacci words in order to obtain a remarkable continued fraction expansion for a well-known constant called the rabbit constant. This project is based on a scholarly paper from the *Proceedings of the American Mathematical Society* by J.L. Davison called "A Series and Its Associated Continued Fraction" [Da].

To define the rabbit constant, let $\alpha = (1+\sqrt{5})/2$. For any real number x, let $\lfloor x \rfloor$ be the greatest integer less than or equal to x. For example, $\lfloor \alpha \rfloor = 1$. We will be interested in the numbers $\lfloor r\alpha \rfloor$, where r is an integer greater than 0. The first few values of $\lfloor r\alpha \rfloor$ are given in the following table.

r	$r\alpha$	$\lfloor r\alpha \rfloor$
1	1.618	1
2	3.2361	3
3	4.5841	4
4	6.4721	6
5	8.0902	8
6	9.7082	9

The rabbit constant $T(\alpha)$ is defined [Da] as the infinite sum

$$T(\alpha) = \sum_{r=1}^{\infty} \frac{1}{2^{\lfloor r\alpha \rfloor}} = \frac{1}{2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^6} + \frac{1}{2^8} + \frac{1}{2^9} + \cdots$$

By comparison with the geometric series, $\sum \frac{1}{2^n}$ we see the infinite sum converges. Taking partial sums, the rabbit constant has the value [Fi]

$$T(\alpha) = .7098034428\ldots$$

In this paper our goal is to show that $T(\alpha)$ can be written as the infinite continued fraction

$$T(\alpha) = \sum_{r=1}^{\infty} \frac{1}{2^{\lfloor r\alpha \rfloor}} = \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{4 + \cfrac{1}{8 + \cfrac{1}{\ldots + \cfrac{1}{t_n + \ldots}}}}}}}$$

where $t_n = 2^{f_{n-1}}$ and $\{f_n\}$ is the Fibonacci sequence. Recall that the Fibonacci sequence (f_n) is defined by $f_0 = 0, f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$. It follows from proving that $T(\alpha)$ is equal to this continued fraction that $T(\alpha)$ is irrational.

2 Continued Fractions

2.1 Finite Continued Fractions

Our discussion of this series and its associated continued fraction begins with the study of continued fractions.

We begin with a definition of finite continued fractions. The following definitions, theorems, and propositions come from *Introduction to Number Theory* [Er] unless otherwise stated.

Definition 1. A *finite continued fraction* is an expression of the form:

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ldots + \cfrac{1}{a_{n-1} + \frac{1}{a_n}}}}} \tag{1}$$

where $a_0, a_1, \ldots, a_n$ are real numbers and $a_1, \ldots, a_n$ are positive. The a_i are the *partial quotients* of the continued fraction. If the partial quotients are all integers, the continued fraction is *simple*. We use the notation $[a_0; a_1, a_2, \ldots, a_n]$ to represent the continued fraction, (1).

Theorem 2. *Any finite simple continued fraction represents a rational number. Conversely, any rational number can be represented by a finite simple continued fraction.*

For example:

$$\begin{aligned}
\frac{40}{31} &= 1 + \frac{9}{31} \\
&= 1 + \frac{1}{31/9} \\
&= 1 + \frac{1}{3 + 4/9} \\
&= 1 + \frac{1}{3 + \frac{1}{9/4}} \\
&= 1 + \cfrac{1}{3 + \cfrac{1}{2 + \frac{1}{4}}}
\end{aligned}$$

2.2 Convergents

We extend our research of continued fractions by looking only at a limited number of terms of the continued fraction. These pieces of continued fractions will help characterize our infinite continued fraction containing the Fibonacci sequence.

Definition 3. For $1 \leq k \leq n$, the k^{th} convergent, C_k, of a continued fraction $[a_0; a_1, \ldots, a_n]$ is the continued fraction

$$C_k = [a_0; a_1, \ldots, a_k].$$

We extend the definition to include $k = 0$, and so $C_0 = a_0$.

Proposition 4. *Let $a_0, a_1, \ldots, a_n$ be real numbers with $a_1, \ldots, a_n$ positive. For the continued fraction $[a_0; a_1, \ldots, a_n]$, define:*

$$\begin{aligned}
p_0 &= a_0 & q_0 &= 1 \\
p_1 &= a_1 p_0 + 1 & q_1 &= a_1 \\
p_k &= a_k p_{k-1} + p_{k-2} & q_k &= a_k q_{k-1} + q_{k-2} \quad (2 \leq k \leq n).
\end{aligned}$$

Then $C_k = p_k/q_k$, *for* $0 \le k \le n$.

Proof. We will use induction on k. Since $C_0 = a_0$ and $C_1 = a_0 + 1/a_1 = (a_0a_1+1)/a_1$, the result holds for $k=0$ and $k=1$. Now assume $n \ge 1$ and that this proposition is true for *any* continued fraction with n terms. Then

$$[a_0; a_1, \dots, a_n]$$

is a continued fraction with $n+1$ partial quotients. Let p_k and q_k be defined as in the statement of the proposition. We need to show that $C_k = \frac{p_k}{q_k}$ for $0 \le k \le n$.
By the induction hypothesis, the proposition holds for the continued fraction

$$[a_0; a_1, \dots, a_{n-1}],$$

that is, $C_k = \frac{p_k}{q_k}$ for $0 \le k \le n-1$. We need to show that $C_n = \frac{p_n}{q_n}$. Notice that

$$\underbrace{[a_0; a_1, \dots, a_{n-1}, a_n]}_{n+1 \text{ terms}} = \left[\underbrace{a_0; a_1, \dots, a_{n-2}, a_{n-1} + \frac{1}{a_n}}_{n \text{ terms}}\right]$$

$$= \frac{(a_{n-1} + \frac{1}{a_n})p_{n-2} + p_{n-3}}{(a_{n-1} + \frac{1}{a_n})q_{n-2} + q_{n-3}}.$$

This last equality holds by the induction hypothesis, since the continued fraction above it has n terms. We can simplify this to obtain

$$\begin{aligned} C_n = [a_0; a_1, \dots, a_n] &= \frac{(a_{n-1}p_{n-2} + p_{n-3}) + \frac{1}{a_n}p_{n-2}}{(a_{n-1}q_{n-2} + q_{n-3}) + \frac{1}{a_n}q_{n-2}} \\ &= \frac{p_{n-1} + \frac{1}{a_n}p_{n-2}}{q_{n-1} + \frac{1}{a_n}q_{n-2}} \\ &= \frac{a_np_{n-1} + p_{n-2}}{a_nq_{n-1} + q_{n-2}} \\ &= \frac{p_n}{q_n} \end{aligned}$$

The second and fourth equalities above follow from the definition of p_k and q_k. □

Theorem 5. *Let* p_k *and* q_k *be as defined in Proposition 4. Then for* $1 \le k \le n$,

$$p_kq_{k-1} - p_{k-1}q_k = (-1)^{k-1}.$$

Proof. We proceed by induction on k. For our base case of $k = 1$, we have

$$p_1q_0 - p_0q_1 = (a_0a_1 + 1) \cdot 1 - a_0a_1 = (-1)^0.$$

Now suppose that

$$p_kq_{k-1} - p_{k-1}q_k = (-1)^{k-1},$$

for some $k \geq 1$. By Proposition 4 we have

$$\begin{aligned} p_{k+1}q_k - p_kq_{k+1} &= (a_{k+1}p_k + p_{k-1})q_k - p_k(a_{k+1}q_k + q_{k-1}) \\ &= -(p_kq_{k-1} - p_{k-1}q_k) \\ &= -(-1)^{k-1}. \end{aligned}$$

This last equality follows from the induction hypothesis. Therefore, $p_{k+1}q_k - p_kq_{k+1} = (-1)^k$. $\square$

Example of Theorem 5

Consider the case for the finite continued fraction $\frac{13}{8}$.

$$\frac{13}{8} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2}}}}$$

In the following table, we illustrate the formula $p_kq_{k-1} - p_{k-1}q_k = (-1)^{k-1}$ for this continued fraction.

1	$\frac{2}{1}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{8}{5}$	$\frac{13}{8}$
$1 \cdot 1 - 1 \cdot 2$	$2 \cdot 2 - 3 \cdot 1$	$3 \cdot 3 - 5 \cdot 2$	$5 \cdot 5 - 8 \cdot 3$	$8 \cdot 8 - 13 \cdot 5$	
$1 - 2$	$4 - 3$	$9 - 10$	$25 - 24$	$64 - 65$	
$= -1$	$= 1$	$= -1$	$= 1$	$= -1$	

Corollary 6. *With notation as in Proposition 4,*

$$\begin{aligned} C_k - C_{k-1} &= \tfrac{(-1)^{k-1}}{q_kq_{k-1}} \quad \text{for } 1 \leq k \leq n, & (2) \\ \text{and } C_k - C_{k-2} &= \tfrac{a_k(-1)^k}{q_kq_{k-2}} \quad \text{for } 1 < k \leq n. & (3) \end{aligned}$$

Proof. By Proposition 4,

$$C_k - C_{k-1} = \frac{p_k}{q_k} - \frac{p_{k-1}}{q_{k-1}} = \frac{p_k q_{k-1} - p_{k-1} q_k}{q_k q_{k-1}} = \frac{(-1)^{k-1}}{q_k q_{k-1}}.$$

The last equality for Equation (2) follows from Theorem 5.

Also by Proposition 4,

$$\begin{aligned}
C_k - C_{k-2} &= \frac{p_k}{q_k} - \frac{p_{k-2}}{q_{k-2}} \\
&= \frac{p_k q_{k-2} - p_{k-2} q_k}{q_k q_{k-2}} \\
&= \frac{(a_k p_{k-1} + p_{k-2})(q_{k-2}) - (p_{k-2})(a_k q_{k-1} + q_{k-2})}{q_k q_{k-2}} \\
&= \frac{(a_k p_{k-1} q_{k-2} + p_{k-2} q_{k-2}) - (a_k p_{k-2} q_{k-1} + p_{k-2} q_{k-2})}{q_k q_{k-2}} \\
&= \frac{a_k p_{k-1} q_{k-2} - a_k p_{k-2} q_{k-1}}{q_k q_{k-2}} \\
&= \frac{a_k (p_{k-1} q_{k-2} - p_{k-2} q_{k-1})}{q_k q_{k-2}} \\
&= \frac{a_k (-1)^{k-2}}{q_k q_{k-2}} \\
&= \frac{a_k (-1)^k}{q_k q_{k-2}}
\end{aligned}$$

The last equalities for Equation (3) follow similarly from Theorem 5, and from the fact that k and $k-2$ have the same parity. ☐

Corollary 7. *Let $C_0, C_1, C_2, \ldots$ be the convergents of a finite continued fraction. Then*

$$C_0 < C_2 < C_4 < \cdots < C_5 < C_3 < C_1.$$

That is, whenever $k \geq 0$ is even and $l \geq 1$ is odd, we have

$$\begin{aligned}
C_k &< C_{k+2}, && (4) \\
C_{l+2} &< C_l, && (5) \\
C_k &< C_l. && (6)
\end{aligned}$$

Proof. If $k \geq 0$ is even, then $k+2$ is even, and so by Equation (3) we have

$$C_{k+2} - C_k = \frac{a_{k+2}(-1)^{k+2}}{q_k q_{k+2}} = \frac{a_{k+2}}{q_k q_{k+2}} > 0.$$

Similarly, if $l \geq 1$ is odd, then $(-1)^{l+2}$ is odd, and

$$C_{l+2} - C_l = \frac{a_{l+2}(-1)^{l+2}}{q_l q_{l+2}} = \frac{-a_{l+2}}{q_l q_{l+2}} < 0$$

and so $C_{l+2} \leq C_l$. Now, by Equation (2) we have that if $k \geq 2$ is even, then

$$C_k - C_{k-1} = \frac{(-1)^{k-1}}{q_k q_{k-1}} = \frac{-1}{q_k q_{k-1}} < 0$$

since $k-1$ is odd, and therefore $C_k < C_{k-1}$ if k is even. Similarly, it can be shown that $C_{k-1} < C_k$ if k is odd.

Now suppose that $k \geq 0$ is even and $l \geq 1$ is odd. Then there are integers i and j such that $k = 2i$ and $l = 2j+1$. Suppose also that $k < l$. Then

$$\begin{aligned} C_l - C_k &= C_{2j+1} - C_{2i} \\ &= (C_{2j+1} - C_{2j}) + (C_{2j} - C_{2j-2}) + (C_{2j-2} - C_{2j-4}) + \ldots + (C_{2i+2} - C_{2i}) \end{aligned}$$

Now, by Equation (4) and since $C_{2j} < C_{2j+1}$, each of the terms in the last sum is positive. Thus $C_l - C_k > 0$.

Similarly, if $k > l$, then

$$C_{2i} - C_{2j+1} = (C_{2i} - C_{2i-1}) + (C_{2i-1} - C_{2i-3}) + (C_{2i-3} - C_{2i-5}) + \ldots + (C_{2j+3} - C_{2j+1}).$$

Again, by Equation (5) and since $C_{2i-1} > C_{2i}$, $C_l - C_k > 0$ since each of the terms in the sum on the right are negative. □

2.3 Infinite Continued Fractions

We shall continue our study of continued fractions for when the fraction does not terminate. Our knowledge of convergents can also be applied to these infinite continued fractions.

Definition 8. From *Continued Fractions* [Ol], we see that an *infinite simple continued fraction* is of the form:

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\cdots}}}}$$

or $x = [a_0; a_1, a_2, a_3, \ldots] = \lim_{n\to\infty}[a_0, a_1, \ldots, a_n]$, where the three dots indicate that the process is continued indefinitely.

Theorem 9. *Let $a_0, a_1, a_2, \ldots$ be an infinite sequence of integers with a_i positive for $i \geq 1$. Let*

$$C_n = [a_0; a_1, a_2, \ldots, a_n].$$

Then the sequence $C_0, C_1, C_2, \ldots$ converges.

Proof. We will show that, given $\varepsilon > 0$, there exists an N such that

$$|C_l - C_k| < \varepsilon,$$

whenever $N \leq k \leq l$. A sequence satisfying this condition is called a *Cauchy sequence* and will necessarily converge to some real number (Theorem 2.6.4 [Ab]). If k is odd, then Corollary 7 implies that

$$C_{k-1} < C_l \leq C_k,$$

whenever $l \geq k$, while if k is even, then the corollary implies that

$$C_k \leq C_l < C_{k-1}$$

if $l \geq k$. In either case,

$$|C_l - C_k| \leq |C_k - C_{k-1}|.$$

Let $p_0, q_0, p_1, q_1, \ldots, p_l, q_l$ be the quantities defined in Proposition 4 for the continued fraction $[a_0; a_1, \ldots, a_l]$. We may conclude, using Corollary 6, that

$$|C_l - C_k| \leq |C_k - C_{k-1}| = \left| \frac{(-1)^{k-1}}{q_k q_{k-1}} \right| = \frac{1}{q_k q_{k-1}}$$

Since a_i are positive integers for $i \geq 1$, the q_i grow arbitrarily large, by the Archmedian Property [Ab], we can find N such that $q_N > 1/\varepsilon$. Now for $N \leq k \leq l$, we have

$$|C_l - C_k| \leq \frac{1}{q_k q_{k-1}} \leq \frac{1}{q_N q_{N-1}} \leq \varepsilon.$$

The second inequality follows since $\{q_i\}$ is an increasing sequence because $a_i \geq 1$ for $i \geq 1$. Thus, if $k \geq N$, then $q_k q_{k-1} > q_N q_{N-1}$ means $\frac{1}{q_k q_{k-1}} \leq \frac{1}{q_N q_{N-1}}$. Hence the sequence is Cauchy, and so the infinite continued fraction converges. □

3 Base Two Representations

In this section we shall look at the base two representation of integers, non-integers, and a theorem about decimal representation in base two.

3.1 Base Two Representation of an Integer

The *base two representation* of an integer m is a string of digits $d_1 d_2 \dots d_k$ where each d_i is either 0 or 1 and the leading digit $d_1 \neq 0$, and

$$m = d_1 \cdot 2^{k-1} + d_2 \cdot 2^{k-2} + \dots + d_{k-1} \cdot 2 + d_k.$$

If m is a base 10 integer, we denote the base two representation of m as $[m]_2$, and write $[m]_2 = d_1 \dots d_k$.

For example: $[53]_2 = 110101$, since $1 \cdot 2^5 + 1 \cdot 2^4 + 0 \cdot 2^3 + 1 \cdot 2^2 + 0 \cdot 2^1 + 1 = 53$.

3.2 Base Two Representation of a Non-integer

Suppose $0 < \alpha < 1$. The base two representation of α is the string $0.a_1 a_2 a_3 \dots$ where a_i is 0 or 1, and $\alpha = \frac{a_1}{2} + \frac{a_2}{2^2} + \frac{a_3}{2^3} + \dots + \frac{a_i}{2^i} + \dots$

For example:
$\left[\frac{1}{3}\right]_2 = .01010101\dots = 0.\overline{01}$ since

$$\begin{aligned}
\frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^6} + \dots &= \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \dots \\
&= \sum_{i=1}^{\infty} \left(\frac{1}{4}\right)^i \\
&= \frac{1}{1 - \frac{1}{4}} - 1 \\
&= \frac{1}{\frac{3}{4}} - 1 \\
&= \frac{4}{3} - 1 \\
&= \frac{1}{3}.
\end{aligned}$$

3.3 Base Two Representation Theorem for Repeating Decimals

Theorem 10. *If $a_1 \dots a_k$ is the base two representation of some integer m, then the fraction*

$$\frac{m}{2^k - 1}$$

has base two decimal representation $0.\overline{a_1 \dots a_k}$.

Proof. Suppose an integer m has a base two representation $a_1a_2\ldots a_k$ where a_i is either 0 or 1, and where $a_1 \neq 0$. Then $m = a_1 \cdot 2^{k-1} + a_2 \cdot 2^{k-2} + \ldots + a_{k-1} \cdot 2 + a_k$. Thus,

$$\begin{aligned}
\frac{m}{2^k-1} &= \frac{m}{2^k} \cdot \frac{1}{1-\frac{1}{2^k}} \\
&= \frac{m}{2^k} \cdot \sum_{i=0}^{\infty} \left(\frac{1}{2^k}\right)^i \\
&= \frac{m}{2^k}\left(1 + \frac{1}{2^k} + \frac{1}{2^{2k}} + \frac{1}{2^{3k}} + \ldots\right) \\
&= \frac{m}{2^k} + \frac{m}{2^{2k}} + \frac{m}{2^{3k}} + \ldots \\
&= \frac{1}{2^k}\left(a_1 \cdot 2^{k-1} + \ldots + a_{k-1} \cdot 2 + a_k\right) + \frac{1}{2^{2k}}\left(a_1 \cdot 2^{k-1} + \ldots + a_{k-1} \cdot 2 + a_k\right) \\
&\quad + \frac{1}{2^{3k}}\left(a_1 \cdot 2^{k-1} + \ldots + a_{k-1} \cdot 2 + a_k\right) + \ldots \\
&= \left(\frac{a_1}{2} + \frac{a_2}{2^2} + \ldots + \frac{a_{k-1}}{2^{k-1}} + \frac{a_k}{2^k}\right) + \left(\frac{a_1}{2^{k+1}} + \frac{a_2}{2^{k+2}} + \ldots + \frac{a_{k-1}}{2^{2k-1}} + \frac{a_k}{2^{2k}}\right) \\
&\quad + \left(\frac{a_1}{2^{2k+1}} + \frac{a_2}{2^{2k+1}} + \ldots + \frac{a_k}{2^{3k}}\right) + \ldots
\end{aligned}$$

Note that this is precisely the number whose base two representation is $0.\overline{a_1a_2\ldots a_k}$. $\square$

4 Fibonacci Word

In this section we will define the Fibonacci word to formed by repeated concatenation of the two previous terms, beginning with the words a and ab, in the same way that the Fibonacci numbers are formed by repeated addition of the two previous terms, beginning with the numbers 0 and 1.

More specifically, let $w_1 = a$, $w_2 = ab$, and for $n \geq 3$, define $w_n = w_{n-1}w_{n-2}$. So

$$\begin{aligned}
w_1 &= a \\
w_2 &= ab \\
w_3 &= \underbrace{ab}\,\underbrace{a} \\
w_4 &= \underbrace{aba}\,\underbrace{ab} \\
w_5 &= \underbrace{abaab}\,\underbrace{aba}
\end{aligned}$$

Continuing this process indefinitely and since w_n is a prefix of w_{n+1}, we arrive at an infinite word

$$w_\infty = abaababaabaab\ldots$$

This infinite word is known as the Fibonacci word. The sequence of lengths of the words w_n is the traditional sequence of Fibonacci numbers. Thus the length of w_k is f_{k+1} digits.

5 Conclusions

We are now ready to return to the problem stated in the introduction. The base two representation of the rabbit constant is the Fibonacci word, with $a = 1$ and $b = 0$. That is, $[T(\alpha)]_2 = 0.1011010110110\ldots$ This was proved in [Pe] as part of an earlier senior capstone project.

First, we will define cf as the Fibonacci continued fraction by

$$cf = \cfrac{1}{1 + \cfrac{1}{2^1 + \cfrac{1}{2^1 + \cfrac{1}{2^2 + \cfrac{1}{2^3 + \cfrac{1}{\cdots}}}}}}$$

We see it can be rewritten as:

$$cf = [0; 2^0, 2^1, 2^1, 2^2, 2^3, 2^5, 2^8, 2^{13}, \ldots]$$

Note the sequence of exponents are $0, 1, 1, 2, 3, 5, 8, 13, \ldots$ and these are Fibonacci numbers. Thus, for $k \geq 1$, the k^{th} partial quotient a_k is $a_k = 2^{f_{k-1}}$.

With this notation the continued fraction is

$$[0; 2^{f_0}, 2^{f_1}, 2^{f_2}, 2^{f_3}, 2^{f_4}, 2^{f_5}, 2^{f_6}, 2^{f_7}, 2^{f_8}, \ldots]$$

$$= [0; 1, 2, 2, 4, 8, 32, 256, 8192, 2097152, 17179869184, \ldots]$$

Next, we will show that $[cf]_2 = 0.1011010110110\ldots$ It will follow then that $cf = T(\alpha)$. Let $\frac{p_k}{q_k}$ be the k^{th} convergent of cf. Now from Proposition 4, we see that

$$p_k = a_k p_{k-1} + p_{k-2} = 2^{f_{k-1}} p_{k-1} + p_{k-2},$$

$$q_k = a_k p_{k-1} + q_{k-2} = 2^{f_{k-1}} q_{k-1} + q_{k-2}.$$

We will show that

$$\frac{p_k}{q_k} = \frac{m_k}{2^{f_{k+1}} - 1}$$

where $[m_k]_2 = w_k$, with $a = 1$ and $b = 0$. For instance, the first few convergents are

$$\begin{aligned}
\frac{p_0}{q_0} &= \frac{0}{1} = \frac{0}{2^{f_1}-1} \qquad \text{and} \quad [0]_2 = 0 = w_0 \\
\frac{p_1}{q_1} &= \frac{1}{1} = \frac{1}{2^{f_2}-1} \qquad \text{and} \quad [1]_2 = 1 = w_1 \\
\frac{p_2}{q_2} &= \frac{1}{1+\frac{1}{2}} = \frac{2}{3}
\end{aligned}$$

So $\left[\frac{2}{3}\right]_2 = \overline{.10}$, by Theorem 10.

$$\begin{aligned}
p_3 &= 2^{f_2} \cdot p_2 + p_1 = 2^1(2) + 1 = 2^2 + 1 = 5 = 101 \\
q_3 &= 2^{f_2} q_2 + q_1 = 2^1(3) + 1 = 7 = 111
\end{aligned}$$

So $\left[\frac{5}{7}\right]_2 = \overline{.101}$, by Theorem 10.

$$\begin{aligned}
p_4 &= 2^{f_3} \cdot p_3 + p_2 = 2^2(2^2+1) + 2 = 2^4 + 2^2 + 2 = 22 = 10110 \\
q_4 &= 2^{f_3} q_3 + q_2 = 2^2(7) + 3 = 31 = 11111
\end{aligned}$$

So $\left[\frac{22}{31}\right]_2 = \overline{.10110}$, by Theorem 10.

We shall now use our past knowledge in the following lemmas about p_k and q_k.

Lemma 11. *p_k is the number whose base two representation is w_k for $k \geq 1$.*

Proof. We shall continue with a proof by induction. We have base cases where $k = 1$ and $k = 2$. We have just seen that $[p_1]_2 = [1]_2 = 1 = w_1$ and $[p_2]_2 = [2]_2 = 10 = w_2$. Now assume $n \geq 2$, and that $[p_k]_2 = w_k$ when $1 \leq k \leq n$. We need to show $[p_{n+1}]_2 = w_{n+1}$. By induction hypothesis, base two representation of p_{n-1} is w_{n-1}, and base two representation of p_n is w_n. Also, since multiplying by 2 adds a zero to the end of the base two representation, the base two representation of $2^{f_n} p_n$ adds f_n 0's to the base two representation of p_n. Thus, since w_{n-1} has f_n digits, our base two representation of $2^{f_n} p_n + p_{n-1}$ is $w_n w_{n-1} = w_{n+1}$. ☐

Lemma 12. *$q_k = 2^{f_{k+1}} - 1$ for $k \geq 1$.*

Proof. We shall continue with a proof by induction. We see that our base cases are as follows: $q_1 = 1 = 2^{f_2} - 1$ and $q_2 = 3 = 2^{f_3} - 1$. Let $n \geq 2$. Assume $q_k = 2^{f_{k+1}} - 1$ for all k with $1 \leq k \leq n$. We need to show $q_{n+1} = 2^{f_{n+2}} - 1$.

$$
\begin{aligned}
q_{n+1} &= a_{n+1}q_n + q_{n-1} && \text{by Proposition 4.} \\
&= 2^{f_n}q_n + q_{n-1} && \text{by definition of } a_n. \\
&= 2^{f_n}(2^{f_{n+1}} - 1) + 2^{f_n} - 1 && \text{by induction hypothesis.} \\
&= 2^{f_n+f_{n+1}} - 2^{f_n} + 2^{f_n} - 1
\end{aligned}
$$

Thus $q_{n+1} = 2^{f_{n+2}} - 1$. □

Finally, from these definitions, theorems, corollaries, and examples combined, we can see that this continued fraction converges to the rabbit constant. To see why, notice

$$\frac{p_k}{q_k} = \frac{p_k}{2^{f_{k+1}} - 1}$$

Consider the base two representation of $\frac{p_k}{q_k}$, $\left[\frac{p_k}{q_k}\right]_2$. By Lemma 11, w_k is the base two representation of p_k, i.e., $w_k = [p_k]_2$. We saw that w_k has length f_{k+1}. So by Theorem 10, $\frac{p_k}{2^{f_{k+1}-1}}$ has base two representation $0.\overline{w_k}$. By definition of the rabbit constant, $\lim_{k\to\infty} 0.\overline{w_k} = 0.w_\infty$, where w_∞ equals the Fibonacci word. We have therefore proved the following theorem.

Theorem 13. *The continued fraction expansion*

$$T(\alpha) = [0; 2^0, 2^1, 2^1, 2^2, 2^3, 2^5, 2^8, 2^{13}, \ldots]$$

whose k^{th} partial quotient is $2^{f_{k-1}}$ converges to the rabbit constant. Thus, the base two expansion of $T(\alpha)$ is the Fibonacci word.

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