# Diophantine Approximations: Examining the Farey Process and its Method on Producing Best Approximations

Kelly Bowden

# 1 Introduction

When a person hears the phrase "irrational number," one does not think of anything clean and definite, especially a mathematician. Irrational numbers are typically estimated to a finite number of decimal places, for example $\pi \approx 3.14159$, even though they have an infinite number of decimal places. There are valid methods that help give us closer approximations to irrational numbers in order to obtain a more exact value. Approximating an irrational number by a rational number is one topic in the field of diophantine approximation [1]. These approximations have been around since Diophantos of Alexandria who lived in A.D. 250; however, diophantine equations, which are polynomial equations such that the variables only represent integer values, have been examined even before Diophantos lived.

One fundamental problem of diophantine approximation is to estimate the value of an irrational number using a rational number of small denominator. Farey fractions are an easily accessible tool for such approximations. Farey fractions were studied by British geologist John Farey Sr. (1766-1826). These fractions specifically help estimate irrational numbers between 0 and 1 using rational fractions in irreducible form [2]. In this paper, we will be studying the mediant, which is the fundamental building block of the Farey process, and its many properties. We will also carefully define the Farey process and how it can be used to estimate irrational numbers by forming a sequence of best left and best right approximations, in addition to using the Farey process in order to determine a *best* approximation to an irrational number. Furthermore, we will end the paper with an application of the Farey process in an educational setting.

First, let us define the terms associated with the Farey process as carefully outlined in the article *Continued Fractions Without Tears* written by Ian Richards [4].

# 2 Mediant Fractions

We will begin with the fundamental definition of the theory behind the Farey process.

**Definition 1.** The **mediant** of a pair of fractions $\frac{a}{b}$ and $\frac{c}{d}$ is the fraction formed by adding the numerators and denominators, that is,

$$\frac{a+c}{b+d}. \tag{1}$$

Theorem 2 shows how the mediant is ordered among $\frac{a}{b}$ and $\frac{c}{d}$.

**Theorem 2.** *Let $a, b, c, d > 0$. If $\frac{a}{b} < \frac{c}{d}$, then*

$$\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}. \tag{2}$$

*Proof.* First, notice that $\frac{c}{d} - \frac{a}{b} > 0$. Thus $\frac{bc-ad}{bd} > 0$, and so $bc - ad > 0$, since $bd > 0$. Now let us first prove that $\frac{a}{b} < \frac{a+c}{b+d}$. We have that

$$\begin{aligned} \frac{a+c}{b+d} - \frac{a}{b} &= \frac{(a+c)b - a(b+d)}{b(b+d)} \\ &= \frac{ab+bc-ab-ad}{b(b+d)} \\ &= \frac{bc-ad}{b(b+d)}. \end{aligned}$$

From our assumption, we know $a, b, c, d > 0$ and $bc - ad > 0$. Thus

$$\frac{bc-ad}{b(b+d)} > 0,$$

and therefore

$$\frac{a+c}{b+d} > \frac{a}{b}.$$

The proof for $\frac{a+c}{b+d} < \frac{c}{d}$ is similar. Therefore $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$. □

# 3 The Farey Process

The mediant has more interesting properties when its adjacent fractions form a Farey pair.

**Definition 3.** A **Farey pair** is a pair of two nonnegative fractions, say $\frac{a}{b}$ and $\frac{c}{d}$, with $0 \leq \frac{a}{b} < \frac{c}{d} \leq 1$ and $bc - ad = 1$. A **Farey interval** is an interval, $[\frac{a}{b}, \frac{c}{d}]$, such that the endpoints are a Farey pair.

Theorem 2 and Definition 3 form the basis of the Farey process. The Farey process essentially starts with a Farey pair $(\frac{a}{b}, \frac{c}{d})$, both of which are irreducible, and takes the mediant, $\frac{a+c}{b+d}$. This in turn creates two new Farey intervals $[\frac{a}{b}, \frac{a+c}{b+d}]$ and $[\frac{a+c}{b+d}, \frac{c}{d}]$. We prove this formally in Theorem 4. The next step in the Farey process is to take the mediant of the Farey pairs $(\frac{a}{b}, \frac{a+c}{b+d})$ and $(\frac{a+c}{b+d}, \frac{c}{d})$, which are

$$\frac{a+(a+c)}{b+(b+d} = \frac{2a+c}{2b+d}$$

and

$$\frac{(a+c)+c}{(b+d)+d} = \frac{a+2c}{b+2d},$$

respectively. One would then continue taking the mediants of each new Farey pair, each time the denominators of the mediant are larger than either of the fractions in the Farey pair. The Farey process was studied in [4]. The following table is an example of the first five rows of the Farey process starting with the Farey pair $(\frac{0}{1}, \frac{1}{1})$.

| **Row** | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | $\frac{0}{1}$ | | | | | | | | | | | | | | | | $\frac{1}{1}$ |
| 2 | $\frac{0}{1}$ | | | | | | | | $\frac{1}{2}$ | | | | | | | | $\frac{1}{1}$ |
| 3 | $\frac{0}{1}$ | | | | $\frac{1}{3}$ | | | | $\frac{1}{2}$ | | | | $\frac{2}{3}$ | | | | $\frac{1}{1}$ |
| 4 | $\frac{0}{1}$ | | $\frac{1}{4}$ | | $\frac{1}{3}$ | | $\frac{2}{5}$ | | $\frac{1}{2}$ | | $\frac{3}{5}$ | | $\frac{2}{3}$ | | $\frac{3}{4}$ | | $\frac{1}{1}$ |
| 5 | $\frac{0}{1}$ | $\frac{1}{5}$ | $\frac{1}{4}$ | $\frac{2}{7}$ | $\frac{1}{3}$ | $\frac{3}{8}$ | $\frac{2}{5}$ | $\frac{3}{7}$ | $\frac{1}{2}$ | $\frac{4}{7}$ | $\frac{3}{5}$ | $\frac{5}{8}$ | $\frac{2}{3}$ | $\frac{5}{7}$ | $\frac{3}{4}$ | $\frac{4}{5}$ | $\frac{1}{1}$ |

Figure 1. We can see from Figure 1 that as the Farey process progresses, the number of fractions in each row increases as a result of forming the mediants between each Farey pair. Several interesting observations can be noted from examining the table. These observations will be generalized for the entire Farey process and proven in the following section.

# 4 Observations

There are several interesting patterns to notice that occur within the Farey process. For any Farey pair $(\frac{a}{b}, \frac{c}{d})$ it is true that

$$\frac{c}{d} - \frac{a}{b} = \frac{bc-ad}{bd} = \frac{1}{bd},$$

[4]. It follows directly from the construction that for any row $n$, the distance between $\frac{0}{1}$ and the first fraction in the row is always $\frac{1}{n}$. Moreover, the difference between any Farey pair in row $n$ is no greater than $\frac{1}{n}$. Additionally, when the mediant is formed, the fraction pairs $(\frac{a}{b}, \frac{a+c}{b+d})$ and $(\frac{a+c}{b+d}, \frac{c}{d})$ are Farey pairs. Thus the subintervals formed from the mediant are also Farey intervals. The following theorem is found in [4].

**Theorem 4.** *If $\frac{a}{b}$ and $\frac{c}{d}$ are a Farey pair, then so are $(\frac{a}{b}, \frac{a+c}{b+d})$ and $(\frac{a+c}{b+d}, \frac{c}{d})$. That is, the two subintervals formed by inserting the mediant into a Farey interval are also Farey intervals.*

*Proof.* Assume $[\frac{a}{b}, \frac{c}{d}]$ is a Farey interval with mediant $\frac{a+c}{b+d}$. We will break the proof into two parts, each examining one subinterval formed.

We will first examine $[\frac{a}{b}, \frac{a+c}{b+d}]$. We already know $\frac{a}{b} < \frac{a+c}{b+d}$. We must prove

$b(a+c)-a(b+d) = 1$. First, note that since $[\frac{a}{b}, \frac{c}{d}]$ is a Farey interval, we know $bc-ad = 1$. Well,

$$\begin{aligned} b(a+c) - a(b+d) &= ab + bc - ab - ad \\ &= bc - ad \\ &= 1. \end{aligned}$$

Thus $[\frac{a}{b}, \frac{a+c}{b+d}]$ is a Farey subinterval.

Now let us examine the subinterval $[\frac{a+c}{b+d}, \frac{c}{d}]$. Again we already know $\frac{a+c}{b+d} < \frac{c}{d}$. We must prove $c(b+d) - d(a+c) = 1$. Similarly,

$$\begin{aligned} c(b+d) - d(a+c) &= bc + cd - ad - cd \\ &= bc - ad \\ &= 1. \end{aligned}$$

Thus $[\frac{a+c}{b+d}, \frac{c}{d}]$ is a Farey subinterval. □

We will now formally prove several of the observations made.

**Lemma 5.** *If $\frac{a_n}{b_n}$ and $\frac{c_n}{d_n}$ are a Farey pair in row $n$, then either $b_n \geq n$ or $d_n \geq n$, and their difference $\frac{c_n}{d_n} - \frac{a_n}{b_n} = \frac{1}{b_n d_n} \leq \frac{1}{n}$.*

*Proof.* We will prove this lemma using induction. Thus, it is true for the first row: $\frac{1}{1} - \frac{0}{1} = 1 \leq 1$. Let $n > 1$ and suppose the lemma is true for the $(n-1)^{th}$ row. That is, if $\frac{a}{b}$ and $\frac{c}{d}$ are adjacent in row $n-1$, then $\frac{c}{d} - \frac{a}{b} \leq \frac{1}{n-1}$ with either $b \geq n-1$ or $d \geq n-1$. Also suppose $\frac{p}{q} < \frac{r}{s}$ are adjacent in row $n$. By how the Farey process is constructed, at least one of $\frac{p}{q}$ and $\frac{r}{s}$ is the mediant of an adjacent Farey pair, say $(\frac{a}{b}, \frac{c}{d})$, from row $n-1$. Without loss of generality, suppose $\frac{r}{s} = \frac{a+c}{b+d}$. Then $\frac{p}{q} = \frac{a}{b}$ and $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$ are adjacent in row $n$. Thus, by Theorem 4,

$$\frac{a+c}{b+d} - \frac{a}{b} = \frac{b(a+c) - a(b+d)}{b(b+d)} = \frac{ab + bc - ab - ad}{b(b+d)} = \frac{bc - ad}{b(b+d)} = \frac{1}{b(b+d)} = \frac{1}{b^2 + bd}.$$

By the induction hypothesis, $\frac{1}{bd} = \frac{c}{d} - \frac{a}{b} < \frac{1}{n-1}$, and either $b \geq n-1$ or $d \geq n-1$, so $b + d \geq (n-1) + 1 = n$ when $n \geq 2$. Also, $bd > n-1$ and we have

$$b^2 + bd > 1 + (n-1) = n.$$

Thus

$$\frac{r}{s} - \frac{p}{q} = \frac{a+c}{b+d} - \frac{a}{b} = \frac{1}{b^2+bd} < \frac{1}{n}.$$

□

As the Farey process progresses, the difference between adjacent Farey fractions approaches zero. Thus, for any real number $x$ such that $0 < x \leq 1$, the Farey process provides a sequence of fractions that approach $x$. We will now prove that an irrational number $0 < \alpha < 1$ is the point of intersection.

**Theorem 6.** *Let $\alpha$ be an irrational number and $0 < \alpha < 1$. Let $(\frac{a_n}{b_n}, \frac{c_n}{d_n})$ be the Farey pair in the $n^{th}$ step of the Farey process closest to $\alpha$, with $\frac{a_n}{b_n} < \alpha < \frac{c_n}{d_n}$. As $n \to \infty$, $\frac{a_n}{b_n} \to \alpha$ and $\frac{c_n}{d_n} \to \alpha$.*

*Proof.* We know either

$$\frac{a_n}{b_n} = \frac{a_{n+1}}{b_{n+1}} \text{ and } \frac{c_{n+1}}{d_{n+1}} < \frac{c_n}{d_n},$$

or

$$\frac{a_n}{b_n} < \frac{a_{n+1}}{b_{n+1}} \text{ and } \frac{c_{n+1}}{d_{n+1}} = \frac{c_n}{d_n}.$$

Either way, $[\frac{a_{n+1}}{b_{n+1}}, \frac{c_{n+1}}{d_{n+1}}] \subseteq [\frac{a_n}{b_n}, \frac{c_n}{d_n}]$. By the nested interval property, there exists an $\alpha \in \bigcap[\frac{a_n}{b_n}, \frac{c_n}{d_n}]$ $\forall n \in \mathbb{N}$ [5]. We know $\frac{c_n}{d_n} - \frac{a_n}{b_n} = \frac{1}{b_n d_n} \leq \frac{1}{n}$, as proved in Lemma 5. Thus as $n \to \infty$, $\frac{c_n}{d_n} - \frac{a_n}{b_n} \to 0$. It then follows that $\alpha$ is the only point in the intersection, so $\frac{a_n}{b_n} \to \alpha$ and $\frac{c_n}{d_n} \to \alpha$ as $n \to \infty$. □

# 5 Best Left and Best Right Approximations

One of the immediate consequences of the definition of the Farey process is that it results in finding best left and best right approximations for an irrational number $\alpha$. The following definition was first given in [4].

**Definition 7.** Let $\alpha$ be an irrational number with $0 < \alpha < 1$. Then a fraction $\frac{p}{q}$ is called a **best left** (respectively, **best right**) approximation to $\alpha$ if:

(i) $\frac{p}{q} < \alpha$ (respectively, $\frac{p}{q} > \alpha$); and

(ii) There is no fraction $\frac{x}{y}$ between $\frac{p}{q}$ and $\alpha$ with a denominator $y \leq q$.

We will now provide more notation to describe the Farey process. For $0 < \alpha < 1$, let $\frac{a_n}{b_n}$ and $\frac{c_n}{d_n}$ be the fractions constructed by the Farey process in the $n^{th}$ step that are closest to

$\alpha$ on the left and right respectively. So we have for $n \geq 1$ that

$$\frac{a_n}{b_n} < \alpha < \frac{c_n}{d_n}.$$

We will show later in Theorem 10 that $\frac{a_n}{b_n}$ and $\frac{c_n}{d_n}$ are the respective best left and best right approximations. Now in row $n+1$ of the Farey process, either

$$\frac{a_{n+1}}{b_{n+1}} = \frac{a_n}{b_n} \text{ and } \frac{c_{n+1}}{d_{n+1}} = \frac{a_n + c_n}{b_n + d_n}, \tag{3}$$

or

$$\frac{a_{n+1}}{b_{n+1}} = \frac{a_n + c_n}{b_n + d_n} \text{ and } \frac{c_{n+1}}{d_{n+1}} = \frac{c_n}{d_n}. \tag{4}$$

Suppose (4) holds, and let $s \geq 1$ be such that $\frac{c_{n+k}}{d_{n+k}} = \frac{c_n}{d_n}$ for $1 \leq k \leq s$ and that $\frac{c_{n+s+1}}{d_{n+s+1}} = \frac{a_{n+s} + c_n}{b_{n+s} + d_n}$. Thus we have the following sequence of approximations to $\alpha$:

$$\frac{a_n}{b_n} < \frac{a_{n+1}}{b_{n+1}} < \frac{a_{n+2}}{b_{n+2}} < .... < \frac{a_{n+s}}{b_{n+s}} < \alpha < \frac{c_{n+s+1}}{d_{n+s+1}} < \frac{c_n}{d_n}.$$

Note that $\frac{a_{n+1}}{b_{n+1}} = \frac{a_n + c_n}{b_n + d_n}$, $\frac{a_{n+2}}{b_{n+2}} = \frac{a_{n+1} + c_n}{b_{n+1} + d_n}$, and $\frac{a_{n+s}}{b_{n+s}} = \frac{a_{n+s-1} + c_n}{b_{n+s-1} + d_n}$. Thus in general

$$\frac{a_{n+k}}{b_{n+k}} = \frac{a_{n+k-1} + c_n}{b_{n+k-1} + d_n},$$

for $1 \leq k \leq s$. Also note that,

$$\frac{a_{n+s} + c_n}{b_{n+s} + d_n} = \frac{c_{n+s-1}}{d_{n+s-1}} > \alpha,$$

and so

$$\frac{a_{n+s+1}}{b_{n+s+1}} = \frac{a_{n+s}}{b_{n+s}}.$$

Now suppose (3) holds. Then there exists an $s \geq 1$ such that, similar to the first case, we will have $\frac{c_{n+k}}{d_{n+k}} = \frac{a_n + c_{n+k-1}}{b_n + d_{n+k-1}}$ and $\frac{a_{n+k}}{b_{n+k}} = \frac{a_n}{b_n}$ for $1 \leq k \leq s$. In this case we have the following sequence of approximations to $\alpha$:

$$\frac{a_n}{b_n} < \frac{a_{n+s+1}}{b_{n+s+1}} < \alpha < \frac{c_{n+s}}{d_{n+s}} < ... < \frac{c_{n+1}}{d_{n+1}} < \frac{c_n}{d_n}.$$

**Example 8.** In order to better illustrate this concept, we will provide an example examining $\frac{a_n}{b_n}$ and $\frac{c_n}{d_n}$ for the irrational number $\alpha = \sqrt{3} - 1 \approxeq .732$.

| $\mathbf{n}$ | $\frac{\mathbf{a_n}}{\mathbf{b_n}}$ | $\frac{\mathbf{c_n}}{\mathbf{d_n}}$ |
|---|---|---|
| 1 | $\frac{0}{1}$ | $\frac{1}{1}$ |
| 2 | $\frac{1}{2}$ | $\frac{1}{1}$ |
| 3 | $\frac{2}{3}$ | $\frac{1}{1}$ |
| 4 | $\frac{2}{3}$ | $\frac{3}{4}$ |
| 5 | $\frac{5}{7}$ | $\frac{3}{4}$ |
| 6 | $\frac{8}{11}$ | $\frac{3}{4}$ |
| 7 | $\frac{8}{11}$ | $\frac{11}{15}$ |
| 8 | $\frac{19}{26}$ | $\frac{11}{15}$ |
| 9 | $\frac{30}{41}$ | $\frac{11}{15}$ |
| 10 | $\frac{30}{41}$ | $\frac{41}{56}$ |

One can see that $\frac{a_n}{b_n}$ keeps getting closer to the value of $\alpha$ in rows 1-3, while $\frac{c_n}{d_n}$ stays at $\frac{1}{1}$. In row 4, $\frac{a_n}{b_n}$ remains the same and $\frac{c_n}{d_n}$ moves closer to $\alpha$. By row 10, our best left approximation is $\frac{30}{41} \approx .7317$ and our best right approximation is $\frac{41}{56} \approx .7321$.

# 6 Demonstrating how the Farey Process Provides Best Left and Best Right Approximations

In order to be a best left or best right approximation, we need to be sure that the $\frac{a_n}{b_n}$ and $\frac{c_n}{d_n}$ are closer to $\alpha$ on the left and right respectively than any fraction of lesser denominator. The following theorem, found in [4], is the key to proving this.

**Theorem 9.** *Among all fractions $\frac{x}{y}$ lying strictly between the Farey pair $(\frac{a}{b}, \frac{c}{d})$, the mediant is the one and only one with the smallest denominator.*

*Proof.* Assume $\frac{x}{y} \neq \frac{a+c}{b+d}$ and $\frac{a}{b} < \frac{x}{y} < \frac{c}{d}$. Note $x, y > 0$.

i) We will first show that $y \geq b+d$. Let $d_1 = \frac{x}{y} - \frac{a}{b} = \frac{bx-ay}{by}$. Since $\frac{x}{y} - \frac{a}{b} > 0$ and $bx - ay > 0$, the numerator $bx - ay \geq 1$ so $d_1 \geq \frac{1}{by}$. Also, let $d_2 = \frac{c}{d} - \frac{x}{y} = \frac{cy-dx}{dy}$. Similarly $cy - dx \geq 1$ so $d_2 \geq \frac{1}{dy}$. Thus

$$d_1 + d_2 \geq \frac{1}{by} + \frac{1}{dy} = \frac{d+b}{bdy} = \frac{1}{bd}\frac{b+d}{y}.$$

Since $(\frac{a}{b}, \frac{c}{d})$ are a Farey pair, we know

$$d_1 + d_2 = \frac{c}{d} - \frac{a}{b} = \frac{bc-ad}{bd} = \frac{1}{bd}.$$

Thus we have the following inequality:

$$\frac{1}{bd} = d_1 + d_2 \geq \frac{1}{bd}\frac{b+d}{y}.$$

Multiplying both sides by $bd$ we have

$$1 \geq \frac{b+d}{y},$$

and therefore $y \geq b+d$.

ii) Now we will show $y > b+d$. We already know the mediant $\frac{a+c}{b+d}$ creates two Farey subintervals. Thus the mediants of these subintervals must have denominators greater than $b+d$. Assume $\frac{x}{y} \in [\frac{a}{b}, \frac{a+c}{b+d}]$. By the previous proof of part one, we can see $y \geq b+(b+d) = 2b+d > b+d$.

Thus $y > b+d$ and the mediant has the smallest denominator. □

The following theorem can be found in [4].

**Theorem 10.** *Let $\alpha$ be an irrational number and let $0 < \alpha < 1$. Let $\frac{a}{b}$ and $\frac{c}{d}$ be fractions given at some step of the Farey process that are closest to $\alpha$ on the left and right respectively. Thus, $\frac{a}{b}$ and $\frac{c}{d}$ are respective best left and best right approximations.*

*Proof.* Consider any Farey pair $\frac{a}{b}$ and $\frac{c}{d}$ that are closest to $\alpha$ on the left and right respectively, given at some step of the Farey process. By Theorem 9 we know all fractions between $\frac{a}{b}$ and $\frac{c}{d}$ have denominators that are greater than both $b$ and $d$. Thus the fractions $\frac{a}{b}$ and $\frac{c}{d}$ are clearly best left and best right approximations to any irrational number $\alpha$ lying between them. □

The following theorem can be found in [4].

**Theorem 11.** *Every rational number* $\frac{p}{q}$ *written in lowest terms, with* $0 < \frac{p}{q} < 1$*, appears at some stage in the Farey process.*

*Proof.* We will show this through a proof by contradiction. Assume there exists a fraction $\frac{p}{q}$, with $0 < \frac{p}{q} < 1$, that never appears in the Farey process. This means at every stage in the Farey process, $\frac{p}{q}$ is somewhere between two adjacent fractions in the process. We know by Theorem 4 that these fractions are always going to be a Farey pair. Let $n \geq 1$, with $n$ being the row number. We know at the $n^{th}$ step, $\frac{a_n}{b_n} < \frac{p}{q} < \frac{c_n}{d_n}$, with $\frac{a_n}{b_n}$ and $\frac{c_n}{d_n}$ being the endpoints of the Farey interval in which $\frac{p}{q}$ is an element. For clarification we are not stating $\frac{p}{q}$ is the mediant. By Lemma 5, we know that at least one of the following is true: $b_n \geq n$ or $d_n \geq n$. By Theorem 9, we have $q \geq b_n + d_n \geq n + 1$. Thus since $n$ is arbitrary, we have a contradiction. □

The following theorem can be found in [4].

**Theorem 12.** *Every best left or best right approximation to an irrational number* $\alpha$*, with* $0 < \alpha < 1$*, appears as the closest fraction to* $\alpha$ *on either the left or right at some step in the Farey process.*

*Proof.* By Theorem 11 we know all fractions written in lowest terms appear at some stage in the Farey process. Thus take any best left or best right approximation, say $\frac{x}{y}$, and examine it the first time it appears in the Farey process. By definition, $\frac{x}{y}$ must be the mediant of its two neighbors, call them $\frac{a}{b}$ and $\frac{c}{d}$. So we have $\frac{a}{b} < \frac{x}{y} < \frac{c}{d}$. Consider the following two cases:

i) Suppose $\frac{a}{b} < \alpha < \frac{c}{d}$. Then either $\alpha \in [\frac{a}{b}, \frac{x}{y}]$ or $\alpha \in [\frac{x}{y}, \frac{c}{d}]$. Either way $\frac{x}{y}$ occurs in the Farey process.

ii) Suppose $\alpha$ is not between $\frac{a}{b}$ and $\frac{c}{d}$. Then $\alpha < \frac{a}{b}$ or $\alpha > \frac{c}{d}$. Either way, one of $\frac{a}{b}$ or $\frac{c}{d}$ is closer to $\alpha$ than $\frac{x}{y}$. However, $\frac{a}{b}$ or $\frac{c}{d}$ have smaller denominators than $\frac{x}{y}$, thus $\frac{x}{y}$ is not a best approximation to $\alpha$. □

# 7 Best Approximation

Obtaining a best left and best right approximation is helpful in estimating irrational numbers, but what if we could obtain a single rational number as a *best approximation* to an irrational number $\alpha$?

**Definition 13.** A rational number $\frac{p}{q}$ is considered a **best approximation** to an irrational number $\alpha$ if and only if $|\alpha - \frac{p}{q}| < |\alpha - \frac{x}{y}|$ whenever $1 \leq y \leq q$.

Notice it follows directly from this definition that every best approximation is either a best left or best right approximation, depending on which side of $\alpha$ the fraction $\frac{p}{q}$ is on. We are now interested in whether the converse is true, that is, whether every best left or best right approximation is a best approximation.

**Example 14.** In order to better illustrate this concept, we will provide an example giving a best approximation to the irrational number $\alpha = e - 2 \approxeq .7182818$. Let $\frac{a_n}{b_n}$ and $\frac{c_n}{d_n}$ denote the respective best left and best right approximations produced by the $n^{th}$ step of the Farey process. The first six values of $\frac{a_n}{b_n}$ and $\frac{c_n}{d_n}$ are provided in the following table. Note the best approximations are bold-faced.

| **n** | $\mathbf{\frac{a_n}{b_n}}$ | $\mathbf{\frac{c_n}{d_n}}$ |
|---|---|---|
| 1 | $\frac{0}{1}$ | $\mathbf{\frac{1}{1}}$ |
| 2 | $\mathbf{\frac{1}{2}}$ | $\frac{1}{1}$ |
| 3 | $\mathbf{\frac{2}{3}}$ | $\frac{1}{1}$ |
| 4 | $\frac{2}{3}$ | $\mathbf{\frac{3}{4}}$ |
| 5 | $\mathbf{\frac{5}{7}}$ | $\frac{3}{4}$ |
| 6 | $\mathbf{\frac{5}{7}}$ | $\frac{8}{11}$ |

One can easily see that $\frac{1}{1}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}$, and $\frac{5}{7}$ are all best approximations. Let us examine the best left approximation of $\frac{5}{7}$ and the best right approximation $\frac{3}{4}$. Clearly $|\alpha - \frac{5}{7}| \approx .003996 < .0317 \approx |\alpha - \frac{3}{4}|$. Thus no fractions of denominator less than 7 exist that are closer to $\alpha$ since all fractions between $\frac{5}{7}$ and $\frac{3}{4}$ have denomiator at least 11. However, when we reach row 6, $\frac{8}{11}$ is the first best right approximation that is not a best approximation because $|\alpha - \frac{5}{7}| \approx .0039 < |\alpha - \frac{8}{11}| \approx .0089$, but $7 < 11$. This now raises the question: What conditions qualify a best one-sided approximation to be a best approximation?

The following theorem takes into consideration the cases where the best approximation is a best left or best right approximation.

**Theorem 15.** *Let* $\frac{a_n}{b_n}$ *and* $\frac{c_n}{d_n}$ *be the sequence of respective best left and best right approximations to* $\alpha$ *produced by the Farey process. For a given* $n$*, let* $s$ *be the largest nonnegative integer for which* $\frac{c_n}{d_n} = \frac{c_{n+s}}{d_{n+s}}$*. Then* $\frac{a_{n+k}}{b_{n+k}}$ *is a best approximation if* $\frac{s}{2} < k \leq s$*. Similarly, if*

*$s$ is the largest nonnegative integer such that $\frac{a_n}{b_n} = \frac{a_{n+s}}{b_{n+s}}$, then $\frac{c_{n+k}}{d_{n+k}}$ is a best approximation if $\frac{s}{2} < k \leq s$.*

*Proof.* We will first consider the case where the best approximation is a best left approximation. Suppose first that $s$ is the largest nonnegative integer such that $\frac{c_n}{d_n} = \frac{c_{n+s}}{d_{n+s}}$. Then the following order is true:

$$\frac{a_n}{b_n} < \frac{a_{n+1}}{b_{n+1}} < ... < \frac{a_{n+k}}{b_{n+k}} < ... < \frac{a_{n+s}}{b_{n+s}} < \alpha < \frac{c_{n+s+1}}{d_{n+s+1}} < \frac{c_n}{d_n}. \tag{5}$$

We are interested in determining when $\frac{a_{n+k}}{b_{n+k}}$ is a best approximation. Since $\frac{a_{n+k}}{b_{n+k}}$ and $\frac{c_n}{d_n}$ form a Farey pair, there are no fractions between $\frac{a_{n+k}}{b_{n+k}}$ and $\frac{c_n}{d_n}$ of denominator less than $b_{n+k}$. Thus, if $|\alpha - \frac{a_{n+k}}{b_{n+k}}| < |\alpha - \frac{c_n}{d_n}|$, then $\frac{a_{n+k}}{b_{n+k}}$ will be a best approximation, since $b_{n+k} = b_{n+k-1} + d_n \geq d_n$. Let $k$ be such that $\frac{s}{2} < k \leq s$. Then we need to determine when

$$\left|\alpha - \frac{a_{n+k}}{b_{n+k}}\right| < \left|\frac{c_{n+s+1}}{d_{n+s+1}} - \frac{a_{n+k}}{b_{n+k}}\right| < \left|\frac{c_{n+s+1}}{d_{n+s+1}} - \frac{c_n}{d_n}\right| < \left|\alpha - \frac{c_n}{d_n}\right|,$$

that is, we want to show $\frac{a_{n+k}}{b_{n+k}}$ is closer to $\alpha$ than $\frac{c_n}{d_n}$.

It is clear from 5 that

$$\left|\alpha - \frac{a_{n+k}}{b_{n+k}}\right| < \left|\frac{c_{n+s+1}}{d_{n+s+1}} - \frac{a_{n+k}}{b_{n+k}}\right|.$$

It is also clear that

$$\left|\frac{c_{n+s+1}}{d_{n+s+1}} - \frac{c_n}{d_n}\right| < \left|\alpha - \frac{c_n}{d_n}\right|.$$

Our goal therefore is to show

$$\left|\frac{c_{n+s+1}}{d_{n+s+1}} - \frac{a_{n+k}}{b_{n+k}}\right| < \left|\frac{c_{n+s+1}}{d_{n+s+1}} - \frac{c_n}{d_n}\right|. \tag{6}$$

Recall,

$$\begin{aligned}
a_{n+1} &= a_n + c_n \\
a_{n+2} &= (a_n + c_n) + c_n = a_n + 2c_n \\
a_{n+3} &= (a_n + 2c_n) + c_n = a_n + 3c_n \\
&\vdots \\
a_{n+k} &= (a_n + (k-1)c_n) + c_n = a_n + kc_n \\
&\vdots \\
a_{n+s} &= (a_n + (s-1)c_n) + c_n = a_n + sc_n
\end{aligned}$$

and

$$\begin{aligned}
b_{n+1} &= b_n + d_n \\
b_{n+2} &= (b_n + d_n) + d_n = b_n + 2d_n \\
b_{n+3} &= (b_n + 2d_n) + d_n = b_n + 3d_n \\
&\vdots \\
b_{n+k} &= (b_n + (k-1)d_n) + d_n = b_n + kd_n \\
&\vdots \\
b_{n+s} &= (b_n + (s-1)d_n) + d_n = b_n + sd_n
\end{aligned}$$

Also, $c_{n+s+1} = a_n + (s+1)c_n$ and $d_{n+s+1} = b_n + (s+1)d_n$. If $1 \le k \le s$, we have the following:

$$\begin{aligned}
\left|\frac{c_{n+s+1}}{d_{n+s+1}} - \frac{a_{n+k}}{b_{n+k}}\right| &= \frac{|b_{n+k}c_{n+s+1} - a_{n+k}d_{n+s+1}|}{b_{n+k}d_{n+s+1}} \\
&= \frac{|(b_n + kd_n)(a_n + (s+1)c_n) - (b_n + (s+1)d_n)(a_n + kc_n)|}{b_{n+k}d_{n+s+1}}.
\end{aligned}$$

Expanding the numerator and simplifying gives us:

$$\begin{aligned}
\left|\frac{c_{n+s+1}}{d_{n+s+1}} - \frac{a_{n+k}}{b_{n+k}}\right| &= \frac{|ka_nd_n + (s+1)b_nc_n - (s+1)a_nd_n - kb_nc_n|}{b_{n+k}d_{n+s+1}} \\
&= \frac{|k(a_nd_n - b_nc_n) + (s+1)(b_nc_n - a_nd_n)|}{b_{n+k}d_{n+s+1}} \\
&= \frac{|k(-1) + (s+1)(1)|}{b_{n+k}d_{n+s+1}} \\
&= \frac{-k + s + 1}{b_{n+k}d_{n+s+1}}. \qquad (7)
\end{aligned}$$

Also,

$$\begin{aligned}
\left|\frac{c_{n+s+1}}{d_{n+s+1}} - \frac{c_n}{d_n}\right| &= \frac{|c_{n+s+1}d_n - c_nd_{n+s+1}|}{d_{n+s+1}d_n} \\
&= \frac{|(a_n + (s+1)c_n)d_n - c_n(b_n + (s+1)d_n)|}{d_{n+s+1}d_n} \\
&= \frac{|a_nd_n + (s+1)c_nd_n - b_nc_n - (s+1)c_nd_n|}{d_{n+s+1}d_n} \\
&= \frac{|a_nd_n - b_nc_n|}{d_{n+s+1}d_n} \\
&= \frac{1}{d_{n+s+1}d_n}. \qquad (8)
\end{aligned}$$

By 7 and 8, to show when 6 holds we must determine when $\frac{-k+s+1}{b_{n+k}d_{n+s+1}} < \frac{1}{d_{n+s+1}d_n}$. This is true if

$$\frac{-k+s+1}{b_{n+k}d_{n+s+1}} < \frac{1}{d_{n+s+1}d_n},$$

multplying both sides by $d_{n+s+1}$ we obtain

$$\frac{-k+s+1}{b_{n+k}} < \frac{1}{d_n}.$$

Multiplying both sides by $b_{n+k}$ we have

$$-k+s+1 < \frac{b_{n+k}}{d_n} = \frac{b_n + kd_n}{d_n},$$

which implies

$$-k+s+1 < \frac{b_n}{d_n} + k.$$

Rearranging gives

$$s+1-\frac{b_n}{d_n} < 2k,$$

and dividing by 2 gives

$$\frac{s}{2}+\frac{1}{2}-\frac{b_n}{2d_n} < k.$$

Since $\frac{b_n}{d_n} < 1$ we have

$$\frac{b_n}{2d_n} < \frac{1}{2},$$

and so

$$0 < \frac{1}{2} - \frac{b_n}{2d_n} < \frac{1}{2}.$$

Thus, if $k > \frac{s}{2}+\frac{1}{2}-\frac{b_n}{2d_n}$, then 6 holds. If $k > \frac{s}{2}$, then $k \geq \frac{s}{2}+\frac{1}{2} > \frac{s}{2}+\frac{1}{2}-\frac{b_n}{2d_n}$ and therefore 6 holds. Thus $\frac{-k+s+1}{b_{n+k}d_{n+s+1}} < \frac{1}{d_{n+s+1}d_n}$ is true if $k > \frac{s}{2}$. So if $\frac{s}{2} < k \leq s$, then $\frac{a_{n+k}}{b_{n+k}}$ is best.

The proof for if $\frac{a_n}{b_n} = \frac{a_{n+s}}{b_{n+s}}$, then $\frac{c_{n+k}}{d_{n+k}}$ is a best approximation is similar.

□

**Example 16.** We will now demonstrate this concept with an example examining the irrational number $\alpha = \frac{-12+2\sqrt{39}}{3} \approx .1633$. The fractions $\frac{a_n}{b_n}$ and $\frac{c_n}{d_n}$ provided in the following table are the respective best left and best right approximations for $\alpha$, while the best approximations are bolded. Additionally, $k$ is provided in order to better illustrate the concept of which best one-sided approximations are best approximations based on the condition of $\frac{s}{2} < k \leq s$.

| $\mathbf{n}$ | $\mathbf{k}$ | $\frac{\mathbf{a_n}}{\mathbf{b_n}}$ | $\frac{\mathbf{c_n}}{\mathbf{d_n}}$ |
|---|---|---|---|
| 1 | 0 | $\frac{\mathbf{0}}{\mathbf{1}}$ | $\frac{1}{1}$ |
| 2 | 1 | $\frac{\mathbf{0}}{\mathbf{1}}$ | $\frac{1}{2}$ |
| 3 | 2 | $\frac{\mathbf{0}}{\mathbf{1}}$ | $\frac{1}{3}$ |
| 4 | 3 | $\frac{0}{1}$ | $\frac{\mathbf{1}}{\mathbf{4}}$ |
| 5 | 4 | $\frac{0}{1}$ | $\frac{\mathbf{1}}{\mathbf{5}}$ |
| 6 | $s = 5$ | $\frac{0}{1}$ | $\frac{\mathbf{1}}{\mathbf{6}}$ |

We can see for $1 \leq n \leq 6$, the mediant of $\frac{a_{n-1}}{b_{n-1}}$ and $\frac{c_{n-1}}{d_{n-1}}$ is on the right of $\alpha$. Thus when $n = 1$, $s = 5$, and $\frac{s}{2} = 2.5$. So for $3 \leq k \leq 5$, we have $\frac{c_{1+k}}{d_{1+k}}$ is a best approximation to $\alpha$ by Theorem 15. Specifically, $\frac{c_4}{d_4}, \frac{c_5}{d_5}, \frac{c_6}{d_6}$ are all best approximations. One can verify by checking $|\alpha - \frac{c_{1+k}}{d_{1+k}}| < |\alpha - \frac{0}{1}|$ for $3 \leq k \leq 5$.

Now we will examine when $\frac{c_{n+s+1}}{d_{n+s+1}} = \frac{c_{n+s}}{d_{n+s}}$ and $\frac{a_{n+s}}{b_{n+s}} = \frac{a_{n+s-1} + c_n}{b_{n+s-1} + d_n}$.

| **n** | **k** | $\mathbf{\frac{a_n}{b_n}}$ | $\mathbf{\frac{c_n}{d_n}}$ |
|---|---|---|---|
| 6 | 0 | $\frac{0}{1}$ | $\mathbf{\frac{1}{6}}$ |
| 7 | 1 | $\frac{1}{7}$ | $\mathbf{\frac{1}{6}}$ |
| 8 | 2 | $\frac{2}{13}$ | $\mathbf{\frac{1}{6}}$ |
| 9 | 3 | $\frac{3}{19}$ | $\mathbf{\frac{1}{6}}$ |
| 10 | 4 | $\mathbf{\frac{4}{25}}$ | $\frac{1}{6}$ |
| 11 | 5 | $\mathbf{\frac{5}{31}}$ | $\frac{1}{6}$ |
| 12 | 6 | $\mathbf{\frac{6}{37}}$ | $\frac{1}{6}$ |
| 13 | 7 | $\mathbf{\frac{7}{43}}$ | $\frac{1}{6}$ |
| 14 | $s = 8$ | $\mathbf{\frac{8}{49}}$ | $\frac{1}{6}$ |

| **n** | **k** | $\mathbf{\frac{a_n}{b_n}}$ | $\mathbf{\frac{c_n}{d_n}}$ |
|---|---|---|---|
| 14 | 0 | $\mathbf{\frac{8}{49}}$ | $\frac{1}{6}$ |
| 15 | 1 | $\mathbf{\frac{8}{49}}$ | $\frac{9}{55}$ |
| ⋮ | ⋮ | ⋮ | ⋮ |

Now we may start the process over with $n = 6$. Then for $7 \leq n \leq 14$, the mediant of $\frac{a_{n-1}}{b_{n-1}}$ and $\frac{c_{n-1}}{d_{n-1}}$ is on the left. Since $s = 8$, $\frac{s}{2} = 4$ and so for $4 \leq k \leq 8$, we have $\frac{a_{6+k}}{b_{6+k}}$ is a best approximation to $\alpha$ by Theorem 15. Specifically, $\frac{a_{10}}{b_{10}}, \frac{a_{11}}{b_{11}}, \frac{a_{12}}{b_{12}}, \frac{a_{13}}{b_{13}}, \frac{a_{14}}{b_{14}}$ are all best approximations. One can verify by checking $|\alpha - \frac{a_{6+k}}{b_{6+k}}| < |\alpha - \frac{1}{6}|$ for $4 \leq k \leq 8$. Note Theorem 15 states $\frac{s}{2} < k \leq s$, while in this example we have $\frac{s}{2} \leq k \leq s$. This is allowed since Theorem 15 is not an if and only if proof. This specific situation is considered as an area for further research.

# 8 Applications for the Classroom

Every student is taught that when adding two fractions together, one cannot just add the numerators and denominators together, one must first find a common denominator between the two fractions at hand. Once the fractions have a common denominator, then they are allowed to add the numerators. Thus the Farey method for adding two fractions together will at first seem foreign to a student; however, the Farey process can be incorporated into lessons within the classroom as a different, but useful method for problem solving.

For example, when a Farey interval is divided into two subintervals when the mediant is formed, the ratio of the length of the two intervals is $\frac{d}{b}$ [2]. Notice,

$$\begin{aligned} \frac{\frac{a+c}{b+d}-\frac{a}{b}}{\frac{c}{d}-\frac{a+c}{b+d}} &= \frac{\frac{ba+bc-ab-ad}{b(b+d)}}{\frac{bc+dc-ad-dc}{d(b+d)}} \\ &= \frac{bc-ad}{b}\frac{d}{bc-ad} \\ &= \frac{d}{b}. \end{aligned}$$

One source discussed various problems that students can solve using the Farey process [2].

**Example 17.** Farmer Fred has chickens and cows, and he knows that they have a total of 11 heads and 28 legs. How many chickens and how many cows does he have [2]?

We know that the ratio between heads and legs for a cow is $\frac{1}{4}$, while the same ratio for a chicken is $\frac{1}{2}$. We know from the problem that the ratio between heads and legs for the total number of chickens and cows that Farmer Fred has is $\frac{11}{28}$, thus this would be considered the mediant of the two fractions we are trying to obtain since $\frac{1}{4} < \frac{11}{28} < \frac{1}{2}$. Note that $\frac{11}{28}$ has to be equal to the ratio of chicken legs to cow legs. Using one of our previous observations that stated the ratio between the length of two adjacent Farey intervals is $\frac{d}{b}$, we can determine just how many chickens and cows Farmer Fred has.

$$\frac{\frac{11}{28}-\frac{1}{4}}{\frac{1}{2}-\frac{11}{28}} = \frac{4}{3}.$$

Thus the ratio of chicken legs to cow legs is $\frac{4}{3}$, so out of the total of 28 legs, $\frac{4}{7}$ of the total are chicken legs and $\frac{3}{7}$ of the total are cow legs. Thus there are 16 chicken legs and 12 cow legs, making Farmer Fred have 8 chickens and 3 cows [2].

This is just one of many problems that educators can apply in the classroom as another approach to problem solving. One could also apply the Farey process to chemistry problems involving finding out the amount one needs of two solutions in order to make one solution of a certain concentration, in addition to problems involving figuring out distance traveled based on how fast a person was going at various points of the trip.

# 9 Conclusion

In this paper on diophantine approximations, we specifically examined the Farey process which uses irreducible fractions between 0 and 1 in order to produce best left and best right approximations for an irrational number $0 < \alpha < 1$. We then further researched the process in order to discover the condition in which a best one-sided approximation is a best approximation.

Areas for further research would include examining if a best approximation could occur under the condition $\frac{s}{2} \leq k \leq s$ when $\frac{s}{2}$ is a whole number. Additionally, it would be interesting to examine if the Farey process properties stay consistent through other intervals of length one, or if any new properties arise. Furthermore, we may compare the Farey process to other methods of estimating irrational numbers using rational numbers to determine which method is more accurate.

# References


[1] *Diophantine Approximation and Abelian Varieties.* Eds. A Dold, B. Eckmann, F. Takens. Berlin: Springer-Verlag, 1993.

[2] Mihaila, Ioana. FAREY SUMS and Understanding RATIOS. *The Mathematics Teacher*,98:3. October 2004, 158-162.

[3] Niven, Ian, Herbert S. Zuckerman, Hugh L. Montgomery. *An Introduction to the Theory of Numbers.* Fifth Ed. New York: John Wiley & Sons, Inc., 1991.

[4] Richards, Ian. Continued Fractions Without Tears. *Mathematics Magazine*,54:4. September 1981, 163-171.

[5] Richardson, Leonard F. *Advanced Calculus: An Introduction to Linear Analysis.* New Jersey: John Wiley & Sons, Inc, 2008.